

OPERATORS OF SUBPRINCIPAL TYPE

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1. INTRODUCTION

We shall consider the solvability for a classical pseudodifferential operator $P \in \Psi_{cl}^m(M)$ on a C^∞ manifold M . This means that P has an expansion $p_m + p_{m-1} + \dots$ where $p_k \in S_{hom}^k$ is homogeneous of degree k , $\forall k$, and $p_m = \sigma(P)$ is the principal symbol of the operator. A pseudodifferential operator is said to be of principal type if the Hamilton vector field H_{p_m} of the principal symbol does not have the radial direction $\xi \cdot \partial_\xi$ on $p_m^{-1}(0)$, in particular $H_{p_m} \neq 0$. We shall consider the case when the principal symbol vanishes of at least second order at an involutive manifold Σ_2 , then P is not of principal type.

P is locally solvable at a compact set $K \subseteq M$ if the equation

$$(1.1) \quad Pu = v$$

has a local solution $u \in \mathcal{D}'(M)$ in a neighborhood of K for any $v \in C^\infty(M)$ in a set of finite codimension. We can also define microlocal solvability of P at any compactly based cone $K \subset T^*M$, see Definition 2.6.

For pseudodifferential operators of principal type, it is known [4] [10] that local solvability is equivalent to condition (Ψ) on the principal symbol, which means that

$$(1.2) \quad \text{Im } ap_m \text{ does not change sign from } - \text{ to } +$$

along the oriented bicharacteristics of $\text{Re } ap_m$

for any $0 \neq a \in C^\infty(T^*M)$. The oriented bicharacteristics are the positive flow-out of the Hamilton vector field $H_{\text{Re } ap_m} \neq 0$ on which $\text{Re } ap_m = 0$, these are also called semibicharacteristics of p_m . Condition (1.2) is invariant under multiplication of p_m with non-vanishing factors, and symplectic changes of variables, thus it is invariant under conjugation of P with elliptic Fourier integral operators. Observe that the sign changes in (1.2) are reversed when taking adjoints, and that it suffices to check (1.2) for some $a \neq 0$ for which $H_{\text{Re } ap} \neq 0$ according to [11, Theorem 26.4.12].

For operators which are not of principal type, the situation is more complicated and the solvability may depend on the lower order terms. When the set Σ_2 where the principal symbol vanishes of second order is involutive, the subprincipal symbol $\sigma_{sub}(P) = p_{m-1}$ is invariantly defined at Σ_2 . In fact, on Σ_2 it is equal to the *refined principal symbol*, see [11, Theorem 18.1.33].

In the case where the principal symbol is real and vanishes of at least second order at the involutive manifold there are several results, mostly in the case when the principal symbol is a product of real symbols of principal type. Then the operator is not solvable

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if the imaginary part of the subprincipal symbol has a sign change of finite order on a bicharacteristics of one the factors of the principal symbol, see [6], [16], [19] and [20].

This necessary condition for solvability has been extended to some cases when the principal symbol is real and vanishes of second order at the involutive manifold. The conditions for solvability then involves the sign changes of the imaginary part of the subprincipal symbol on the limits of bicharacteristics from outside the manifold, thus on the leaves of the symplectic foliation of the manifold, see [12], [13], [14] and [23]. This has been extended to more general limit bicharacteristics of real principal symbols in [5].

When Σ_2 is not involutive, there are examples where the operator is solvable for any lower order terms. For example when P is effectively hyperbolic, then even the Cauchy problem is solvable for any lower order term, see [9] and [15]. There are also results in the cases when the principal symbol is a product of principal type symbols not satisfying condition (Ψ) , see [1], [7], [8], [17] and [22].

In the present paper, we shall consider the case when the principal symbol (not necessarily real valued) vanishes of at least second order at a non-radial involutive manifold Σ_2 . We shall assume that the subprincipal symbol is of principal type with Hamilton vector field tangent to Σ_2 at the characteristics, but transversal to the symplectic leaves of Σ_2 . We shall also assume that the subprincipal symbol is essentially constant on the symplectic leaves of Σ_2 by (2.8), and does not satisfying condition (Ψ) , see Definition 2.4. In the case when the sign change is of infinite order, we will need a condition on the rate of vanishing of both the Hessian of the principal symbol and the complex part of the gradient of the subprincipal symbol on the semibicharacteristic of the subprincipal symbol, see condition (2.11). Under these conditions, P is not solvable in a neighborhood of the semibicharacteristic, see Theorem 2.7 which is the main result of the paper. In this case P is an evolution operator, see [2] and [3] for some earlier results on the solvability of evolution operators.

2. STATEMENT OF RESULTS

Let $\sigma(P) = p \in S_{hom}^m$ be the homogeneous principal symbol, we shall assume that

$$(2.1) \quad \sigma(P) \text{ vanishes of at least second order at } \Sigma_2$$

where

$$(2.2) \quad \Sigma_2 \text{ is a non-radial involutive manifold}$$

Here non-radial means that the radial direction $\langle \xi, \partial_\xi \rangle$ is not in the span of the Hamilton vector fields of the manifold, i.e., not equal to H_f on Σ_2 for some $f \in C^1$ vanishing at Σ_2 . Then by a change of homogeneous symplectic coordinates we may assume that

$$(2.3) \quad \Sigma_2 = \{ \xi' = 0 \} \quad \xi = (\xi', \xi'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}$$

for some $k > 0$, this can be achieved by conjugation by elliptic Fourier integral operators. If P is of principal type near Σ_2 then, since solvability is an open property, we find that a necessary condition for P to be solvable at Σ_2 is that condition (Ψ) for the principal symbol is satisfied in some neighborhood of Σ_2 . Naturally, this condition is empty on Σ_2

where we instead need some conditions on the subprincipal symbol

$$(2.4) \quad p_s = p_{m-1} + \frac{i}{2} \sum_j \partial_{x_j} \partial_{\xi_j} p$$

which is equal to p_{m-1} on Σ_2 and invariantly defined as a function on Σ_2 under symplectic changes of coordinates and conjugation with elliptic pseudodifferential operators. (In the Weyl quantization, the subprincipal symbol is equal to p_{m-1} .) When composing P with an elliptic pseudodifferential operator C , the value of the subprincipal symbol of CP is equal to $cp_s + \frac{i}{2} H_p c = cp_s$ at Σ_2 , where $c = \sigma(C)$. Observe that the subprincipal symbol is complexly conjugated when taking the adjoint of the operator.

Let $T\Sigma_2^\sigma$ be the symplectic polar to $T\Sigma_2$, which spans the symplectic leaves of Σ_2 . If $\Sigma_2 = \{\xi' = 0\}$, $x = (x', x'') \in \mathbf{R}^k \times \mathbf{R}^{n-k}$, then the leaves are spanned by $\partial_{x'}$. Let

$$(2.5) \quad T^\sigma \Sigma_2 = T\Sigma_2 / T\Sigma_2^\sigma$$

which is a symplectic space over Σ_2 which in these coordinates is given by

$$(2.6) \quad T^\sigma \Sigma_2 = \{((x_0, 0, \xi_0''); (0, y'', 0, \eta'')) : (y'', \eta'') \in T^* \mathbf{R}^{n-k}\}$$

Next, we are going to study the Hamilton vector field $H_{p_{m-1}}$ at Σ_2 . If $H_{p_{m-1}} \subseteq T\Sigma_2$ at Σ_2 then we find that dp_s vanishes on $T\Sigma_2^\sigma$ so dp_s is well defined on $T^\sigma \Sigma_2$. In fact, $p_s = p_{m-1}$ on Σ_2 so if we choose coordinates so that (2.3) holds, then $H_{p_{m-1}} \subseteq T\Sigma_2$ is equivalent to

$$(2.7) \quad H_{p_{m-1}} \xi' = -\partial_{x'} p_{m-1} = -\partial_{x'} p_s = 0 \quad \text{when } \xi' = 0$$

which is invariant under multiplication with non-vanishing factors when $p_s = 0$. Let H_{p_s} be the Hamilton vector field of p_s with respect the symplectic structure on the symplectic manifold $T^\sigma \Sigma_2$. In the chosen coordinates we have

$$H_{p_s} = \partial_{\xi''} p_s \partial_{x''} - \partial_{x''} p_s \partial_{\xi''}$$

modulo $\partial_{x'}$, which is non-vanishing if $\partial_{x'' \xi''} p_s \neq 0$. Since $p_s = p_{m-1}$ on Σ_2 , the difference between $H_{p_{m-1}}$ and H_{p_s} is tangent to the leaves of Σ_2 . Actually, since the subprincipal symbol is only well defined on Σ_2 , the vector field H_{p_s} is only well-defined up to terms tangent to the leaves.

Because of that, we would need that the subprincipal symbol p_s is constant on the leaves of Σ_2 , but that condition is *not* invariant under multiplication with non-vanishing factors when $p_s \neq 0$. Instead we shall use the following invariant condition:

$$(2.8) \quad |dp_s|_{TL} \leq C_0 |p_s|$$

for any leaf L of Σ_2 . Then p_s is constant on the leaves modulo non-vanishing factors, according to the following lemma.

Lemma 2.1. *If $dp_s|_{T^\sigma \Sigma_2} \neq 0$, then condition (2.8) is equivalent to the fact that p_s is constant on the leaves of Σ_2 after multiplication with a smooth non-vanishing factor. Thus, if $\Sigma_2 = \{\xi' = 0\}$ then (2.8) gives $p_s(x, 0, \xi'') = c(x, \xi'') q(x'', \xi'')$ with $0 \neq c \in C^\infty$.*

Proof. Choose coordinates so that $\Sigma_2 = \{\xi' = 0\}$. If $p_s \neq 0$ at a point $w_0 \in \Sigma_2$ then (2.8) gives that $\partial_{x'} \log p_s$ is uniformly bounded near w_0 , where $\log p_s$ is a branch of the complex

logarithm. Thus, by integrating with respect to x' in a simply connected neighborhood starting at $x' = x'_0$ we find that

$$(2.9) \quad p_s(x, 0, \xi'') = c(x, \xi'')q(x'', \xi'')$$

where $q(x'', \xi'') = p_s(x'_0, x'', 0, \xi'') \in C^\infty$ so $0 \neq c \in C^\infty$ satisfies $c(x'_0, x'', \xi'') = 1$. When $p_s = 0$ we find that $d \operatorname{Re} z p_s|_{T^\sigma \Sigma_2} \neq 0$ for some $z \in \mathbf{C} \setminus 0$ by assumption. Thus we obtain locally that

$$p_s(x, 0, \xi'') = c_\pm(x, \xi'')q_\pm(x'', \xi'') \quad \text{on } S_\pm = \{ \pm \operatorname{Re} z p_s(x, 0, \xi'') > 0 \}$$

where $q_\pm(x'', \xi'') = p_s(x'_0, x'', 0, \xi'')$, $0 \neq c_\pm \in C^\infty$ and $c_\pm(x'_0, x'', \xi'') = 1$ on S_\pm . Then we find that $p_s^{-1}(0)$ is independent of x' and

$$\partial_{x''}^\alpha \partial_{\xi''}^\beta q_\pm(x'', \xi'') = \partial_{x''}^\alpha \partial_{\xi''}^\beta p_s(x'_0, x'', 0, \xi'') \quad \forall \alpha \beta \quad \text{on } S_\pm$$

so by taking the limit at $S = \{ \operatorname{Re} z p_s = 0 \}$ we find that the functions q_\pm extend to $q \in C^\infty$. Since $c_+ q = c_- q = p_s$ at S we find that $c_+ = c_-$ at S when $q \neq 0$. When $q = 0$ at S we may differentiate in the normal direction of S to obtain that $c_- \partial_\nu q = c_+ \partial_\nu q$ and since $\partial_\nu q \neq 0$ the functions c_\pm extend to a continuous function c . By differentiating and taking the limit we find that

$$\nabla c_- q + c \nabla q = \nabla p_s = \nabla c_+ q + c \nabla q \quad \text{at } S$$

which similarly gives that $\nabla c_- = \nabla c_+$ at S so $c \in C^1$. By repeatedly differentiating $c_\pm q$ we find by induction that $c \in C^\infty$, so we get smooth quotients c and q in (2.9). \square

Now, a semibicharacteristic of p_s will be a bicharacteristic of $\operatorname{Re} a p_s$ on $T^\sigma \Sigma_2$, $C^\infty \ni a \neq 0$, with the natural orientation. Observe that condition (2.7) is only invariant under multiplication with non-vanishing factors when $p_s = 0$.

Definition 2.2. We say that the operator P is of *subprincipal type* if the following hold when $p_s = 0$ on Σ_2 : $H_{p_{m-1}}|_{\Sigma_2} \subseteq T\Sigma_2$,

$$(2.10) \quad dp_s|_{T^\sigma \Sigma_2} \neq 0$$

and the corresponding Hamilton vector field H_{p_s} of (2.10) does not have the radial direction. We call H_{p_s} the *subprincipal Hamilton vector field* and the (semi)bicharacteristics are called the *subprincipal (semi)bicharacteristics* on Σ_2 .

Clearly, if (2.3) holds, then the condition that the Hamilton vector field does not have the radial direction means that $\partial_{\xi''} p_s \neq 0$ or $\partial_{x''} p_s \not\parallel \xi''$ when $p_s = 0$ on $\Sigma_2 = \{ \xi' = 0 \}$.

In the case when the principal symbol p is real, then a necessary condition for solvability of the operator is that the imaginary part of the subprincipal symbol does not change sign from $-$ to $+$ when going in the positive direction on a C^∞ limit of normalized bicharacteristics of the principal symbol p at Σ_2 , see [5]. When p vanishes of exactly second order on $\Sigma_2 = \{ \xi' = 0 \}$ such limit bicharacteristics are tangent to the leaves of Σ_2 . In fact, then Taylor's formula gives $H_p = \langle B \xi', \partial_{x'} \rangle + \mathcal{O}(|\xi'|^2)$ where $B \neq 0$, so the normalized Hamilton vector fields have limits that are tangent to the leaves. When the principal symbol is proportional to a real valued symbol, this gives examples of non-solvability when the subprincipal symbol is not constant on the leaves of Σ_2 . Thus condition (2.8) is essential if there are no other conditions on the principal symbol.

Remark 2.3. If p_s is real valued, then by the proof of Lemma 2.1 it follows from (2.8) that p_s has constant sign on the leaves of Σ_2 , since then $c > 0$ in (2.9).

Definition 2.4. We say that P satisfies condition $\text{Sub}(\Psi)$ if $\text{Im } ap_s$ does not change sign from $-$ to $+$ when going in the positive direction on the subprincipal bicharacteristics of $\text{Re } ap_s$ for any $0 \neq a \in C^\infty$.

Thus, condition $\text{Sub}(\Psi)$ is condition (Ψ) given by (1.2) on the subprincipal symbol p_s . Observe that since p_s is only defined on Σ_2 , the Hamilton vector field H_{p_s} is only well defined in $T^\sigma \Sigma_2 = T\Sigma_2/T\Sigma_2^\sigma$, thus it is well defined modulo $\partial_{x'}$. But if (2.8) holds then we find that $\text{Sub}(\Psi)$ is a condition on p_s with respect to the symplectic structure of $T^\sigma \Sigma_2$. In fact, by the invariance of condition (Ψ) given by [11, Lemma 26.4.10] condition $\text{Sub}(\Psi)$ holds for any $a \neq 0$ such that $H_{\text{Re } ap_s} \neq 0$, so we may assume by Lemma 2.1 that p_s is constant on the leaves of Σ_2 .

Since condition $\text{Sub}(\Psi)$ is invariant under symplectic changes of variables and multiplication with non-vanishing functions, it is invariant under conjugation of the operator by elliptic Fourier integral operators. Observe that the sign change is reversed when taking the adjoint of the operator.

Recall that the Hessian of the principal symbol $\text{Hess } p$ is the quadratic form given by $d^2 p$ at Σ_2 which is defined on the normal bundle $N\Sigma_2$, since it vanishes on $T\Sigma_2$. By the calculus $\text{Hess } p$ is invariant, modulo non-vanishing smooth factors, under symplectic changes of variables and multiplication of P with elliptic pseudodifferential operators.

Next, we assume that condition $\text{Sub}(\Psi)$ is not satisfied on a semibicharacteristic Γ of p_s , i.e., $\text{Im } ap_s$ changes sign from $-$ to $+$ on the positive flow-out of $H_{\text{Re } ap_s} \neq 0$ for some $0 \neq a \in C^\infty$. Now if the sign change is *not* of finite order, we shall also need an extra condition on the rate of vanishing of both the Hessian of the principal symbol and the complex part of the gradient of the subprincipal symbol on the subprincipal semibicharacteristic. Then, we shall assume that there exists $C > 0$, $\varepsilon > 0$ and $0 \neq a \in C^\infty$ so that $d \text{Re } ap_s|_{T\Sigma_2} \neq 0$ and

$$(2.11) \quad \|\text{Hess } p\| + |dp_s \wedge d\bar{p}_s| \leq C|p_s|^\varepsilon \quad \text{when } \text{Re } ap_s = 0 \text{ on } \Sigma_2$$

near Γ . Since (2.11) also holds for smaller ε and larger C , it is no restriction to assume $\varepsilon \leq 1$. The motivation for (2.11) is to prevent the transport equation (6.1) to disperse the support of the solution before the sign change of the imaginary part of the subprincipal symbol localizes it, see Remark 3.1. We also find that ∇p_s is proportional to a real vector when $p_s = 0$ since then $dp_s \wedge d\bar{p}_s = 0$. Then a is well-defined up to real factors.

Remark 2.5. Condition (2.11) is invariant under multiplication of P with elliptic pseudodifferential operators and symplectic changes of coordinates. If (2.8) also holds, then we obtain that

$$(2.12) \quad \|d \text{Hess } p|_{TL}\| \leq C_1 |p_s|^{\varepsilon/2}$$

for any leaf L of Σ_2 when $\text{Re } ap_s = 0$ near Γ .

In fact, multiplication with an elliptic pseudodifferential operator with principal symbol c changes the principal symbol into cp , the Hessian of the principal symbol into $c \text{Hess } p$

and the subprincipal symbol into

$$cp_s + \frac{i}{2}H_p c \quad \text{at } \Sigma_2$$

where the last term vanishes at Σ_2 and contains the factor $\text{Hess } p$, modulo terms vanishing of second order at Σ_2 . Now we have that

$$|dcp_s \wedge d\overline{cp_s}| \leq |c|^2 |dp_s \wedge d\overline{p_s}| + C|p_s|$$

Thus we find that (2.11) holds with p replaced by cp , p_s replaced by cp_s and a replaced with a/c . If (2.8) also holds and we choose coordinates so that $\Sigma_2 = \{\xi' = 0\}$, then we obtain from Lemma 2.1 that $|p_s(x', x'', 0, \xi'')| \cong |p_s(x'_0, x'', 0, \xi'')|$ when $|x' - x'_0| \leq c$. Thus (2.11) gives

$$\|\text{Hess } p(x', x'', 0, \xi'')\| \leq C_2 |p_s(x'_0, x'', 0, \xi'')|^\varepsilon \quad \text{when } |x' - x'_0| \leq c$$

To show (2.12) it suffices to consider an element $b_{jk}(x', x'', 0, \xi'')$ of $\text{Hess } p$. Clearly $|b_{jk}| \leq \|\text{Hess } p\|$ so by adding $C_2 |p_s(x_0, x'', 0, \xi'')|^\varepsilon$, we obtain that

$$0 \leq b_{jk}(x', x'', 0, \xi'') \leq 2C_2 |p_s(x_0, x'', 0, \xi'')|^\varepsilon \quad \text{when } |x' - x'_0| \leq c$$

Then we find that

$$|\partial_{x'} b_{jk}(x_0, x'', 0, \xi'')| \leq C \sqrt{b_{jk}(x_0, x'', 0, \xi'')} \leq C' |p_s(x_0, x'', 0, \xi'')|^{\varepsilon/2}$$

by [11, Lemma 7.7.2].

We shall study the microlocal solvability of the operator, which is given by the following definition. Recall that $H_{(s)}^{loc}(X)$ is the set of distributions that are locally in the L^2 Sobolev space $H_{(s)}(X)$.

Definition 2.6. If $K \subset S^*X$ is a compact set, then we say that P is microlocally solvable at K if there exists an integer N so that for every $f \in H_{(N)}^{loc}(X)$ there exists $u \in \mathcal{D}'(X)$ such that $K \cap \text{WF}(Pu - f) = \emptyset$.

Observe that solvability at a compact set $M \subset X$ is equivalent to solvability at $S^*X|_M$ by [11, Theorem 26.4.2], and that solvability at a set implies solvability at a subset. Also, by [11, Proposition 26.4.4] the microlocal solvability is invariant under conjugation by elliptic Fourier integral operators and multiplication by elliptic pseudodifferential operators. We can now state the main result of the paper.

Theorem 2.7. *Assume that $P \in \Psi_{cl}^m(X)$ has principal symbol that vanishes of at least second order at a non-radial involutive manifold, is of subprincipal type, does not satisfy condition $\text{Sub}(\Psi)$ on the subprincipal semibicharacteristic $\Gamma \subset T^*X$, and satisfies (2.8) near Γ . In the case the sign change in $\text{Sub}(\Psi)$ is of infinite order we also assume condition (2.11) near Γ . Then P is not locally solvable at Γ .*

Example 2.8. Let

$$(2.13) \quad P = D_1 D_2 + B(x, D_x)$$

with $B \in \Psi_{cl}^1$, then $\sigma(B)$ is the subprincipal symbol on $\Sigma_2 = \{\xi_1 = \xi_2 = 0\}$. Mendoza and Uhlmann proved in [13] that P was not solvable if $\text{Im } \sigma(B)$ changed sign as x_1 or x_2 increases on Σ_2 , and they proved in [12] that P was solvable if $\text{Im } \sigma(B) \neq 0$ on Σ_2 . From

this it is natural to conjecture that the condition for solvability of P is that $\text{Im } \sigma(B)$ does not change sign on the leaves of Σ , which are foliated by ∂_{x_1} and ∂_{x_2} . But the following is a counterexample to that conjecture. Let

$$(2.14) \quad P = D_1 D_2 + D_t + i f(t, x, D_x)$$

with real and homogeneous $f(t, x, \xi) \in S_{hom}^1$ satisfying $\partial_{x_j} f = \mathcal{O}(|f|)$ for $j = 1, 2$. This operator is of subprincipal type and satisfies (2.8). Then Theorem 2.7 gives that P is not solvable if $t \mapsto f(t, x, \xi)$ changes sign of finite order from $-$ to $+$, but observe that f has constant sign on the leaves of Σ_2 by Remark 2.3. Thus the solvability of the operator P in (2.13) also depends on the real part of the subprincipal symbol at Σ_2 . In fact, with the above conditions one can prove that $D_1 D_2 + i f(t, x, D_x)$ is solvable.

Example 2.9. The linearized Navier-Stokes equation

$$(2.15) \quad \partial_t u + \sum_j a_j(t, x) \partial_{x_j} u + \Delta_x u = f \quad a_j(x) \in C^\infty$$

is of subprincipal type. The symbol is

$$(2.16) \quad i\tau + i \sum_j a_j(t, x) \xi_j - |\xi|^2$$

so the subprincipal symbol is proportional to a real symbol on $\Sigma_2 = \{\xi = 0\}$. Thus condition $\text{Sub}(\Psi)$ is satisfied.

Now let $S^*M \subset T^*M$ be the cosphere bundle where $|\xi| = 1$, and let $\|u\|_{(k)}$ be the L^2 Sobolev norm of order k , $u \in C_0^\infty$. In the following, P^* will be the L^2 adjoint of P . To prove Theorem 2.7 we shall use the following result.

Remark 2.10. If P is microlocally solvable at $\Gamma \subset S^*\mathbf{R}^n$, then Lemma 26.4.5 in [11] gives that for any $Y \Subset \mathbf{R}^n$ such that $\Gamma \subset S^*Y$ there exists an integer ν and a pseudodifferential operator A so that $\text{WF}(A) \cap \Gamma = \emptyset$ and

$$(2.17) \quad \|u\|_{(-N)} \leq C(\|P^*u\|_{(\nu)} + \|u\|_{(-N-n)} + \|Au\|_{(0)}) \quad u \in C_0^\infty(Y)$$

where N is given by Definition 2.6.

We shall prove Theorem 2.7 in Section 8 by constructing localized approximate solutions to $P^*u \cong 0$ and use (2.17) to show that P is not microlocally solvable at Γ . We shall first find a normal form for the adjoint operator.

3. THE NORMAL FORM

Assume that P^* has the symbol expansion $p_m + p_{m-1} + \dots$ where $p_j \in S_{hom}^j$ is homogeneous of degree j . By multiplying P^* with an elliptic pseudodifferential operator, we may assume that $m = 2$. Choose local symplectic coordinates $(t, x, y, \tau, \xi, \eta)$ so that $\Sigma_2 = \{\eta = 0\}$, which is foliated by leaves spanned by ∂_y . Since p_2 vanishes of at least second order at Σ_2 we find that

$$p_2(t, x, y, \tau, \xi, \eta) = \sum_{jk} B_{jk}(t, x, y, \tau, \xi, \eta) \eta_j \eta_k$$

where B_{jk} is homogeneous of degree 0, $\forall jk$.

The differential inequality (2.8) in these coordinates means that $|\partial_y p_1| \leq C|p_1|$ when $\eta = 0$, which by Lemma 2.1 gives that

$$p_1(t, x, y, \tau, \xi, 0) = q(t, x, y, \tau, \xi) r_1(t, x, \tau, \xi)$$

near Γ , where q is a non-vanishing smooth homogeneous function. By multiplying with a pseudodifferential operators with principal symbol equal to q^{-1} on Σ_2 we may assume that $q \equiv 1$ and that p_1 is constant on the leaves of Σ_2 near Γ . The Hamilton vector field of p_1 is then tangent to Σ_2 by (2.7).

We have assumed that P does not satisfy condition $\text{Sub}(\Psi)$ on a semibicharacteristic Γ of p_1 on Σ_2 . Since we are now considering the adjoint P^* this means that $\text{Im } ap_1$ changes sign from $+$ to $-$ on the flow-out Γ of $H_{\text{Re } ap_1}$ on $\text{Re } ap_1^{-1}(0)$ for some $0 \neq a \in C^\infty$. By the invariance of condition $\text{Sub}(\Psi)$ given by [11, Lemma 26.4.10], it is no restriction to assume that a is homogeneous and constant in y . By multiplication with an elliptic pseudodifferential operator having principal symbol a^{-1} we may assume that $a \equiv 1$. Since $\text{Im } p_1$ changes sign on Γ there is a maximal semibicharacteristic $\Gamma' \subset \Gamma$ on which $\text{Im } p_1 = 0$. Here Γ' could be a point, which is always the case if the sign change is of finite order.

Since P is of subprincipal type we find that $\partial_{t,x,\tau,\xi} \text{Re } p_1 \neq 0$ on Γ' by (2.10) so Γ' is transversal to the leaves of Σ_2 . Since $\text{Im } p_1|_{\Gamma'}$ has opposite signs near the boundary of Γ' , we may shrink Γ so that it is not a closed curve. Since $H_{\text{Re } p_1}$ is tangent to Σ_2 we can complete $\tau = \text{Re } p_1$ to a symplectic coordinate system in a convex neighborhood of Γ' so that $\eta = 0$ on Σ_2 . In fact, this is obtained by solving the equation $H_\tau \eta = 0$ with initial value on a submanifold transversal to H_τ . The change of variables can be then done by conjugation with suitable elliptic Fourier integral operators.

Now, by using Malgrange's preparation theorem in a neighborhood of Γ' in Σ_2 we find that

$$p_1(t, x, y, \tau, \xi, 0) = q(t, x, \tau, \xi)(\tau + r(t, x, \xi)) \quad q \neq 0$$

near Γ , since p_1 is constant on the leaves of Σ_2 . In fact, on Γ' we have that $p_1 = 0$ and $dp_1 \neq 0$, so the division can be done locally and by a partition of unity globally near Γ after possibly shrinking Γ . Then by using Taylor's formula on p_1 we find since $q \neq 0$ that

$$(3.1) \quad p_1(t, x, y, \tau, \xi, \eta) = q(t, x, \tau, \xi) (\tau + r(t, x, \xi) + A(t, x, y, \tau, \xi, \eta) \cdot \eta)$$

By multiplying P with an elliptic pseudodifferential operator, we may again assume $q \equiv 1$. Since p_2 vanishes of second order at Σ_2 , this only changes A with terms which has $\text{Hess } p_2$ as a factor and terms that vanish at Σ_2 .

We can write $r = r_1 + ir_2$ and $A = A_1 + iA_2$ with real valued r_j and A_j , $j = 1, 2$. Now we may complete

$$\text{Re } p_1 = \tau + r_1(t, x, \xi) + A_1(t, x, y, \tau, \xi, \eta) \cdot \eta$$

to a symplectic coordinate system in a convex neighborhood of Γ' . Since $H_{\text{Re } p_1} \in T\Sigma_2$ at Σ_2 we may keep $\Sigma_2 = \{\eta = 0\}$, which preserves the leaves of Σ_2 on which p_1 is constant. Thus, we find that

$$(3.2) \quad p_1 = \tau + if(t, x, \xi) + iA(t, x, y, \tau, \xi, \eta) \cdot \eta$$

where $f = r_2$ and $A = A_2$ are real valued. We also find that

$$(3.3) \quad \Gamma = \{ (t, x_0, y_0, 0, \xi_0, 0) \} \quad t \in I$$

where I is an interval in \mathbf{R} . The symplectic change of coordinates can be made by conjugation with elliptic Fourier integral operators, which only changes A with terms having $\text{Hess } p_2$ as a factor and terms that vanish at Σ_2 . Observe that A need not be real valued after these changes.

We have assumed that condition $\text{Sub}(\Psi)$ is *not* satisfied for P on the subprincipal semibicharacteristic Γ . Thus the imaginary part of the subprincipal symbol of P^* on Σ_2

$$(3.4) \quad t \mapsto f(t, x_0, \xi_0)$$

changes sign from $+$ to $-$ as t increases on $I \subset \mathbf{R}$. Similarly, we have that $f = 0$ on Γ' where Γ' is given by (3.3) with I replaced by $I' \subset I$. By reducing to *minimal bicharacteristics* on which $t \mapsto f(t, x, \xi)$ changes sign as in [10, p. 75], we may assume that f vanishes of infinite order on a bicharacteristic Γ' arbitrarily close to the original bicharacteristic, if Γ' is not a point (see [21, Section 2] for a more refined analysis). If Γ' is not a point then it is a *one-dimensional bicharacteristic* by [10, Definition 3.5], which means that the Hamilton vector field on Γ' is proportional to a real vector.

In fact, if $f(a, x_0, \xi_0) > 0 > f(b, x_0, \xi_0)$ for some $a < b$, then we can define

$$L(x, \xi) = \inf \{ t - s : a < s < t < b \text{ such that } f(s, x, \xi) > 0 > f(t, x, \xi) \}$$

when (x, ξ) is close to (x_0, ξ_0) , and we put $L_0 = \liminf_{(x, \xi) \rightarrow (x_0, \xi_0)} L(x, \xi)$. Then for every $\varepsilon > 0$ there exists an open neighborhood V_ε of (x_0, ξ_0) such that the diameter of V_ε is less than ε and $L(x, \xi) > L_0 - \varepsilon/2$ when $(x, \xi) \in V_\varepsilon$. By definition, there exists $(x_\varepsilon, \xi_\varepsilon) \in V_\varepsilon$ and $a < s_\varepsilon < t_\varepsilon < b$ so that $t_\varepsilon - s_\varepsilon < L_0 + \varepsilon/2$ and $f(s_\varepsilon, x_\varepsilon, \xi_\varepsilon) > 0 > f(t_\varepsilon, x_\varepsilon, \xi_\varepsilon)$. Then it is easy to see that

$$(3.5) \quad \partial_x^\alpha \partial_\xi^\beta f(t, x_\varepsilon, \xi_\varepsilon) = 0 \quad \forall \alpha, \beta \quad \text{when } s_\varepsilon + \varepsilon < t < t_\varepsilon - \varepsilon$$

since else we would have a sign change in a smaller interval than $L_0 - \varepsilon/2$ in V_ε . We may choose a sequence $\varepsilon_j \rightarrow 0$ so that $s_{\varepsilon_j} \rightarrow s_0$ and $t_{\varepsilon_j} \rightarrow t_0$, then $L_0 = t_0 - s_0$ and (3.5) holds at (x_0, ξ_0) for $s_0 < t < t_0$.

We also obtain the following condition from (2.11).

Remark 3.1. If the sign change of $t \mapsto f(t, x, \xi)$ is of infinite order on Γ , then we find from assumption (2.11) that

$$(3.6) \quad \| \{ B_{jk} \}_{jk} \| + |A| + |df| \lesssim |f|^\varepsilon \quad \text{near } \Gamma \text{ on } \Sigma_2$$

for some $\varepsilon > 0$. Here $a \lesssim b$ (and $b \gtrsim a$) means that $a \leq Cb$ for some $C > 0$.

In fact, terms having $\text{Hess } p_2|_{\Sigma_2} = \{ B_{jk} \}_{jk}$ as a factor can be estimated by (2.11), so we may assume that (3.2) holds with real A . The subprincipal symbol is equal to $p_s = p_1 + i \sum_{jk} \partial_{y_j} B_{jk} \eta_k$ modulo terms that are $\mathcal{O}(|\eta|^2)$ so $p_s = p_1$ on Σ_2 . By Remark 2.5 and (2.8) we can estimate the terms $\partial_{y_j} B_{jk} d\eta_k$ in dp_s by replacing ε with $\varepsilon/2$ in (2.11), so we may replace p_s by p_1 in the estimate. Let $0 \neq a = a_1 + ia_2$ with real valued a_j in (2.11), so that $d \text{Re } ap_1|_{T\Sigma_2} \neq 0$. We have $dp_1 = d\tau + i(df + Ad\eta)$ on Σ_2 so

$$|dp_1 \wedge d\bar{p}_1| \cong |df| + |A| \quad \text{on } \Sigma_2$$

Thus we find from (2.11) that $|df| + |A| = 0$ on Γ' . Since $d \operatorname{Re} ap_1|_{T\Sigma_2} \neq 0$ we find that $a_1 \neq 0$ on Γ' . On Σ_2 we have that $\operatorname{Re} ap_1 = a_1\tau - a_2f = 0$ when $\tau = a_2f/a_1$. We obtain that

$$\operatorname{Im} ap_1 = a_2\tau + a_1f = |a|^2 f/a_1 \quad \text{when } \operatorname{Re} ap_1 = 0 \text{ on } \Sigma_2 \text{ near } \Gamma'$$

which gives (3.6) from (2.11).

We obtain the following normal form for these operators of subprincipal type:

$$(3.7) \quad P^* = D_t + F(t, x, y, D_t, D_x, D_y)$$

where $F \sim F_2 + F_1 + \dots$ with homogeneous $F_j \in C^\infty(\mathbf{R}, S_{hom}^j)$. Here F_2 vanishes of at least second order on $\Sigma_2 = \{\eta = 0\}$, so we find by Taylor's formula that

$$(3.8) \quad F_2(t, x, y, \tau, \xi, \eta) = B(t, x, y, \tau, \xi, \eta) = \sum_{jk} B_{jk}(t, x, y, \tau, \xi, \eta) \eta_j \eta_k$$

with homogeneous B_{jk} , then $\{B_{jk}\}_{jk}|_{\Sigma_2} = \operatorname{Hess} F_2(t, x, y, \tau, \xi, 0)$. Also we have that F_1 vanishes on the semibicharacteristic Γ' and

$$(3.9) \quad F_1(t, x, y, \tau, \xi, \eta) = if(t, x, \xi) + A(t, x, y, \tau, \xi, \eta) \cdot \eta$$

Here f is real and homogeneous of degree 1 and $A|_{\Sigma_2} = \partial_\eta F_1|_{\Sigma_2}$. We have that the principal symbol $\sigma(P^*) = F_2$, and the subprincipal symbol $\sigma_{sub}(P^*) = \tau + if$ on Σ_2 . Thus we obtain the following result.

Proposition 3.2. *Assume that P satisfies the conditions in Theorem 2.7. Then by conjugation with elliptic Fourier integral operators and multiplication with an elliptic pseudodifferential operator we may assume that*

$$(3.10) \quad P^* = D_t + F(t, x, y, D_t, D_x, D_y)$$

microlocally near $\Gamma = \{(t, x_0, y_0, 0, \xi_0, 0) : t \in I\} \subset \Sigma_2$ where $S_{cl}^2 \ni F \cong F_2 + F_1 + \dots$ with $F_j \in S_{hom}^j$ is homogeneous of degree j and

$$F_2 = \sum_{jk} B_{jk}(t, x, y, \tau, \xi, \eta) \eta_j \eta_k \in S_{hom}^2$$

vanishes of second order on Σ_2 . We may also assume that

$$F_1(t, x, y, \tau, \xi, \eta) = if(t, x, \xi) + A(t, x, y, \tau, \xi, \eta) \cdot \eta$$

is homogeneous of degree 1 and f is real valued such that $t \mapsto f(t, x_0, \xi_0)$ changes sign from $+$ to $-$ as t increases on $I \subset \mathbf{R}$. If $f(t, x_0, \xi_0) = 0$ on a subinterval $I' \subseteq I$ such that $|I'| \neq 0$, then we may assume that $\partial_t^k \partial_x^\alpha \partial_\xi^\beta f(t, x_0, \xi_0) = 0$, $\forall k \alpha \beta$, for $t \in I'$. If the sign change of f is of infinite order then (3.6) is satisfied near Γ .

For the proof of Theorem 2.7 we shall modify the Moyer-Hörmander construction of approximate solutions of the type

$$(3.11) \quad u_\lambda(t, x, y) = e^{i\lambda\omega(t, x, y)} \sum_{j \geq 0} \phi_j(t, x, y) \lambda^{-j/N} \quad \lambda \geq 1$$

with N to be determined later. Here the phase function $\omega(t, x)$ will be complex valued, but $\text{Im } \omega \geq 0$ and $\partial \text{Re } \omega \neq 0$ when $\text{Im } \omega = 0$. Letting $z = (t, x, y)$ we therefore have the formal expansion

$$(3.12) \quad p(z, D)(\exp(i\lambda\omega)\phi) \sim \exp(i\lambda\omega) \sum_{\alpha} \partial_{\xi}^{\alpha} p(z, \lambda \partial_z \omega(z)) \mathcal{R}_{\alpha}(\omega, \lambda, D) \phi(z) / \alpha!$$

where $\mathcal{R}_{\alpha}(\omega, \lambda, D)\phi(z) = D_w^{\alpha}(\exp(i\lambda\tilde{\omega}(z, w))\phi(w))|_{w=z}$ and

$$\tilde{\omega}(z, w) = \omega(w) - \omega(z) + (z - w)\partial\omega(z)$$

Observe that the values of the symbol are given by an almost analytic extension, see Theorem 3.1 in Chapter VI and Chapter X:4 in [18]. This gives

$$(3.13) \quad e^{-i\lambda\omega} P^* e^{i\lambda\omega} \phi = \\ (\lambda \partial_t \omega_{\lambda} + \lambda^2 B(t, x, y, \partial_{t,x,y} \omega) + i\lambda f(t, x, \partial_x \omega) - \lambda \partial_{\eta}^2 B(t, x, y, \partial_{t,x,y} \omega) \partial_y^2 \omega / 2) \phi \\ + D_t \phi + \lambda \partial_{\eta} B(t, x, y, \partial_{t,x,y} \omega) D_y \phi + \partial_{\eta}^2 B(t, x, y, \partial_{t,x,y} \omega) D_y^2 \phi / 2 \\ + i \partial_{\xi} f(t, x, \partial_x \omega) D_x \phi + A(t, x, y, \partial_{t,x,y} \omega) D_y \phi + \sum_{j \geq 0} \lambda^{-j} R_j(t, x, y, D_{t,x,y}) \phi$$

where $R_0(t, x, y) = F_0(t, x, y, \partial_{t,x,y} \omega)$. Here the values of the symbols at $(t, x, y, \partial_{t,x,y} \omega)$ will be replaced by finite Taylor expansions at $(t, x, y, \partial_{t,x,y} \text{Re } \omega)$. In fact, the almost analytic extensions are determined by these Taylor expansions.

Because of the inhomogeneity coming from the terms of B , we shall use a phase function $\omega(t, x)$ which is constant in y , so that

$$(3.14) \quad u_{\lambda}(t, x, y) = e^{i\lambda\omega(t,x)} \sum_{j \geq 0} \phi_j(t, x, y) \lambda^{-j/N} \quad \lambda \geq 1$$

When $\partial_y \omega \equiv 0$ the expansion (3.13) becomes

$$(3.15) \quad e^{-i\lambda\omega} P^* e^{i\lambda\omega} \phi = (\lambda(\partial_t \omega_{\lambda} + i f(t, x, \partial_x \omega)) \phi \\ + D_t \phi + \partial_{\eta}^2 B(t, x, y, \partial_{t,x} \omega, 0) D_y^2 \phi / 2 + A(t, x, y, \partial_{t,x} \omega, 0) D_y \phi + i \partial_{\xi} f(t, x, \partial_x \omega) D_x \phi \\ + \sum_{j \geq 0} \lambda^{-j} R_j(t, x, y, D_{t,x,y}) \phi$$

where $R_0(t, x, y) = F_0(t, x, y, \partial_{t,x} \omega, 0)$, and $R_m(t, x, y, D_{t,x,y})$ are differential operators of order j in t , order k in x and order ℓ in y , where $j + k + \ell \leq m + 2$ for $m > 0$. In fact, this follows since $\partial_{\tau}^j \partial_{\xi}^{\alpha} \partial_{\eta}^{\beta} F_k \in S^{k-j-|\alpha|-|\beta|}$ by homogeneity.

4. THE EIKONAL EQUATION

We shall first solve the eikonal equation approximately, which is given by the highest order term of (3.15)

$$(4.1) \quad \partial_t \omega + i f(t, x, \partial_x \omega) = 0$$

where $t \mapsto f(t, x, \xi)$ changes sign from $+$ to $-$ for some (x, ξ) as t increases in a neighborhood of $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$ on which $f(t, x, \xi)$ vanishes. If $|I| \neq 0$ then by reducing to minimal bicharacteristics as in Section 3, we may assume that f vanishes of infinite order at Γ . We shall choose the phase function so that $\text{Im } \omega \geq 0$ and $\partial_x^2 \text{Im } \omega > 0$ near the interval. By changing coordinates, it is no restriction to assume $0 \in I$. We shall use

the approach by Hörmander [10] in the principal type case and use the phase function to localize in t and x . Observe that since ω does not depend on y the localization in the y variables will be done in the amplitude ϕ .

We shall take the Taylor expansion of ω in x :

$$(4.2) \quad \omega(t, x) = w_0(t) + \langle x - x_0(t), \xi_0(t) \rangle + \sum_{2 \leq |\alpha| \leq K} w_\alpha(t) (x - x_0(t))^\alpha / \alpha!$$

Here $\alpha = (\alpha_1, \alpha_2, \dots)$, with $\alpha_j \in \mathbb{N}$, $\alpha! = \prod_j \alpha_j!$ and $|\alpha| = \alpha_1 + \alpha_2 + \dots$. Then we find that

$$(4.3) \quad \begin{aligned} \partial_t \omega(t, x) &= w'_0(t) - \langle x'_0(t), \xi_0(t) \rangle + \langle x - x_0(t), \xi'_0(t) \rangle + \sum_{2 \leq |\alpha| \leq K} w'_\alpha(t) (x - x_0(t))^\alpha / \alpha! \\ &\quad - \sum_{\substack{1 \leq |\alpha| \leq K-1 \\ k}} w_{\alpha+e_k}(t) (x - x_0(t))^\alpha x'_{0,k}(t) / \alpha! \end{aligned}$$

where $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ is the k :th unit vector. We also find

$$(4.4) \quad \partial_{x_j} \omega(t, x) = \xi_{0j}(t) + \sum_{1 \leq |\alpha| \leq K-1} w_{\alpha+e_j}(t) (x - x_0(t))^\alpha / \alpha! = \xi_{0j}(t) + \sigma_j(t, x)$$

Here $\xi_0(t) = (\xi_{0,1}(t), \dots)$ and $\sigma = \{\sigma_j\}_j$ is a finite expansion in powers of $\Delta x = x - x_0$. We define the value of $f(t, x, \partial_x \omega)$ by the Taylor expansion

$$(4.5) \quad \begin{aligned} f(t, x, \partial_x \omega) &= f(t, x, \xi_0 + \sigma) \\ &= f(t, x, \xi_0) + \sum_j \partial_{\xi_j} f(t, x, \xi_0) \sigma_j + \sum_{jk} \partial_{\xi_j} \partial_{\xi_k} f(t, x, \xi_0) \sigma_j \sigma_k / 2 + \dots \end{aligned}$$

Now the value at $x = x_0$ of (4.1) is equal to $w'_0(t) - \langle x'_0(t), \xi_0(t) \rangle + i f(t, x_0(t), \xi_0(t))$. This vanishes if

$$(4.6) \quad \begin{cases} \operatorname{Re} w'_0(t) = \langle x'_0(t), \xi_0(t) \rangle \\ \operatorname{Im} w'_0(t) = -f(t, x_0(t), \xi_0(t)) \end{cases}$$

so by putting $w_0(0) = 0$ this will determine w_0 once we have $(x_0(t), \xi_0(t))$.

We shall simplify the notation and put $w_k = \{w_\alpha k! / \alpha!\}_{|\alpha|=k}$ so that w_k is a multilinear form. The first order terms in $x - x_0$ of (4.1) vanish if

$$\xi'_0(t) - w_2(t) x'_0(t) + i(\partial_x f(t, x_0(t), \xi_0(t)) + \partial_\xi f(t, x_0(t), \xi_0(t)) w_2(t)) = 0$$

We find by taking real and imaginary parts that

$$(4.7) \quad \begin{cases} \xi'_0 = \operatorname{Re} w_2 x'_0 + \partial_\xi f(t, x_0, \xi_0) \operatorname{Im} w_2 \\ x'_0 = (\operatorname{Im} w_2)^{-1} (\partial_x f(t, x_0, \xi_0) + \partial_\xi f(t, x_0, \xi_0) \operatorname{Re} w_2) \end{cases}$$

with $(x_0(0), \xi_0(0)) = (x_0, \xi_0)$, which will determine $x_0(t)$ and $\xi_0(t)$ if $|\operatorname{Im} w_2| \neq 0$.

The second order terms in $x - x_0$ vanish if

$$w'_2/2 - w_3 x'_0/2 + i(\partial_\xi f w_3/2 + \partial_x^2 f/2 + \partial_x \partial_\xi f w_2 + w_2 \partial_\xi^2 f w_2/2) = 0$$

which gives

$$(4.8) \quad w'_2 = w_3 x'_0 - i(\partial_\xi f w_3 + \partial_x^2 f + 2\partial_x \partial_\xi f w_2 + w_2 \partial_\xi^2 f w_2)$$

with initial data $w_2(0)$ so that $\operatorname{Im} w_2(0) > 0$.

We find that the terms of order $k > 2$ vanish if

$$(4.9) \quad w'_k - w_{k+1}x'_0 = F_k(t, x_0, \xi_0, \{w_j\})$$

where we may choose $w_k(0) = 0$. Here F_k is a linear combination of the derivatives of f of order $\leq k$ multiplied by polynomials in w_j with $2 \leq j \leq k+1$. When $k = K$ we get $w'_K = F_K(t, x_0, \xi_0, \{w_j\})$ where $j \leq K$. The equations (4.7)–(4.9) form a quasilinear system of differential equations, which can be solved in a convex neighborhood of 0. In the case when $|I| \neq 0$ we have assumed that $\partial_{t,x,\xi}^\alpha f(t, x_0, \xi_0) \equiv 0$, $\forall \alpha$, for $t \in I$. Then we find from (4.7)–(4.9) that x_0 , ξ_0 and w_k are constant in $t \in I$, so we may solve (4.7)–(4.9) in a convex neighborhood of I . Observe that the lower order terms cannot change the condition that $\text{Im} \partial_x^2 \omega \geq c > 0$ and $\text{Im} \omega(t, x) \geq 0$ if $|x - x_0(t)| \ll 1$. Summing up, we have proved the following result.

Proposition 4.1. *Let $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$ and assume that $\partial_t^k \partial_x^\alpha \partial_\xi^\beta f(t, x_0, \xi_0) = 0$ for all $t \in I$ in the case $|I| \neq 0$. Then we may solve (4.1) with $\omega(t, x)$ given by (4.2) in a convex neighborhood Ω of Γ modulo $\mathcal{O}(|x - x_0(t)|^M)$, $\forall M$, such that $(x_0(t), \xi_0(t)) = (x_0, \xi_0)$ when $t \in I$ and $w_k(t) \in C^\infty$ such that $w_0(t) = 0$, $\text{Im} w_2(t) > 0$ and $w_k(t) = 0$, $k > 2$, when $t \in I$.*

Then we obtain that $\text{Im} \omega(t, x) \geq c|x - x_0(t)|^2$ near Γ , $c > 0$, so the errors that are $\mathcal{O}(|x - x_0|^M)$ in the eikonal equation will give terms that are bounded by $C_M \lambda^{-M/2}$. But we have to show that $t \mapsto f(t, x_0(t), \xi_0(t))$ also changes sign from $+$ to $-$ as t increases for some choice of (x_0, ξ_0) . This problem will be studied in the next section, with a special emphasis on the finite vanishing case. By (4.6) we then obtain that $t \mapsto \text{Im} w_0(t)$ has a local minimum on I which can be equal to 0 by subtracting a constant.

5. THE CHANGE OF SIGN

We have assumed that condition $\text{Sub}(\Psi)$ for P is *not* satisfied near the subprincipal semibicharacteristic $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$, so that $t \mapsto f(t, x, \xi)$ changes sign from $+$ to $-$ for some (x, ξ) as t increases near Γ . But after solving the eikonal equation we have to know that $t \mapsto f(t, x_0(t), \xi_0(t))$ has the same sign change, possibly after changing the starting point (x_0, ξ_0) . In order to do so we shall use the invariance of condition $\text{Sub}(\Psi)$, but note that condition (3.6) is only assumed when the change of sign is of infinite order. Therefore we shall first consider the case when the sign change is of finite order and show that this condition is preserved after solving the eikonal equation. Thus assume that

$$(5.1) \quad \partial_t^k f(t_0, x_0, \xi_0) < 0 \quad \text{and} \quad \partial_t^j f(t_0, x_0, \xi_0) = 0 \quad \text{for } j < k$$

for some odd integer k , where we may assume $t_0 = 0$. Now, if the order of the zero is not constant in a neighborhood of (x_0, ξ_0) then in any neighborhood the mapping $t \mapsto f(t, x, \xi)$ must have a zero of odd order with sign change from $+$ to $-$, and the order of vanishing is constant almost everywhere on $f^{-1}(0)$. In fact, this follows since $\partial_t^k f \neq 0$, $t \mapsto f(t, x, \xi)$ goes from $+$ to $-$ and the set where the order of the zero changes is nowhere dense in $f^{-1}(0)$ since it is the union of boundaries of closed sets in the relative topology. By possibly changing (t_0, x_0, ξ_0) we may assume that (5.1) holds with $t_0 = 0$, and that the order of the zero is odd and constant near (x_0, ξ_0) , then the zeros forms a smooth manifold by the implicit function theorem. By using Taylor's formula, we find

that $f(t, w) = a(t, w)(t - t_0(w))^k$, where $k \geq 1$ is odd, $w = (x, \xi)$, $t_0(w_0) = 0$ and $a < 0$ in a neighborhood of $w_0 = (x_0, \xi_0)$. Then we find

$$(5.2) \quad \partial_w f = \partial_w a(t - t_0)^k - ak(t - t_0)^{k-1} \partial_w t_0$$

which vanishes of at least order $k - 1$ in t at $f^{-1}(0)$. Let $w(t) = (x_0(t), \xi_0(t))$ then

$$f(t, w(t)) = f(t, w_0) + \partial_w f(t, w_0) \Delta w(t) + \mathcal{O}(|\Delta w(t)|^2)$$

where $\Delta w(t) = w(t) - w_0$. Now $t \mapsto f(t, w_0)$ vanishes of order k in t at 0 and $t \mapsto \partial_w f(t, w_0)$ vanishes of at least order $k - 1$, so if $t \mapsto \Delta w(t)$ vanishes of at least order k then by (5.2) we find that $t \mapsto f(t, w(t))$ vanishes of order k . Since $\frac{d}{dt} \Delta w(t) = w'(t)$, we will need the following result.

Lemma 5.1. *Let $(x_0(t), \xi_0(t))$ be the solution to the equation (4.7) with $\text{Im } w_2(0) \neq 0$ and assume that $t \mapsto \partial_w f(t, x_0, \xi_0)$ vanishes of order $r \geq 1$ at $t = 0$. Then $(x'_0(t), \xi'_0(t))$ vanishes of order r and $\Delta w(t)$ vanishes of order $r + 1$ at $t = 0$.*

Proof. By (4.7) we have that

$$(5.3) \quad w'(t) = (x'_0(t), \xi'_0(t)) = A(t) \partial_w f(t, w(t)) \quad w(0) = w_0$$

Here we have $|A(0)| \neq 0$ if $\text{Im } w_2(0) \neq 0$, in fact $w'(0) = 0$ then gives $\partial_\xi f(0, w_0) = 0$ and $\partial_x f(0, w_0) = 0$ by (4.7).

Now we denote $\phi_0(t) = \partial_w f(t, w_0)$ and $\phi_1(t) = \partial_w f(t, w(t))$. Then we have that $w'(t) = A(t) \phi_1(t)$ and the condition is that $\phi_0(t)$ vanishes of order $r \geq 1$ at 0. We shall proceed by induction, and first assume that $r = 1$. Since $w(0) = w_0$ we find $\phi_1(0) = \phi_0(0) = 0$ and thus $w'(0) = 0$.

Next, for $r > 1$ we assume by induction that $w'(t)$ vanishes by order $r - 1$ at 0 so $w^{(k)}(0) = 0$ for $k < r$, and then we shall show that $w^{(r)}(0) = 0$ so that w' vanishes of order r . By using the chain rule we obtain that

$$(5.4) \quad \partial_t^r (g(t, w(t))) = \sum_{\substack{0 \leq j \leq r \\ \sum_i r_i + j = r}} c_{j, \alpha} \partial_t^j \partial_w^\alpha g(t, w(t)) \prod_{i=1}^{|\alpha|} w^{(r_i)}(t)$$

for any $g(t, w) \in C^\infty$. Thus, for $g = \partial_w f$ we find that

$$\phi_1^{(k)}(0) = \phi_0^{(k)}(0) + \partial_t^{k-1} \partial_w^2 f(0, x_0, \xi_0) w'(0) + \cdots + \partial_w^2 f(0, x_0, \xi_0) w^{(k)}(0) = \phi_0^{(k)}(0) = 0$$

for $k < r$, since the other terms has some factor $w^{(j)}(0) = 0$, $j \leq k$, which implies that $\phi_1(t)$ vanishes for order r . Since $w' = A \phi_1$ we find that $w'(t)$ vanishes of order r , which gives the induction step and the proof. \square

Now, if $f(t, w_0)$ vanishes of order k then $\partial_w f(t, w_0)$ vanishes of order $k - 1$. Thus $w'(t)$ vanishes of order $k - 1$ by Lemma 5.1 and since $w(0) = w_0$ we find that $\Delta w(t)$ vanishes of order k . Thus, we find that $f(t, w(t)) - f(t, w_0)$ vanishes of order $2k - 1$, so these terms vanish of same order if $k > 1$. In the case $k = 1$, we shall use an argument of Hörmander [10] for the principal type case. We obtain from (4.6) that $\partial_t (f(t, w(t))) = -\text{Im } w_0''(t)$ thus

$$(5.5) \quad \text{Im } w_0''(0) = -\partial_t f(0, w_0) - \partial_\xi f(0, w_0) \cdot \xi'_0 - \partial_x f(0, w_0) \cdot x'_0$$

where $\partial_t f(0, w_0) = -c < 0$. We find from (4.7) that

$$(5.6) \quad \begin{cases} \xi'_0(0) = \operatorname{Re} w_2(0) x'_0(0) + \partial_\xi f(0, w_0) \operatorname{Im} w_2(0) \\ x'_0(0) = (\operatorname{Im} w_2(0))^{-1} (\partial_x f(0, w_0) + \partial_\xi f(0, w_0) \operatorname{Re} w_2(0)) \end{cases}$$

If $\partial_\xi f(0, w_0) = 0$ then we find that $x'_0(0) = (\operatorname{Im} w_2(0))^{-1} \partial_x f(0, w_0)$ and obtain

$$(5.7) \quad \operatorname{Im} w''_0(0) = c - \partial_x f(0, w_0) (\operatorname{Im} w_2(0))^{-1} \partial_x f(0, w_0) > c/2 > 0$$

by choosing $\operatorname{Im} w_2(0) = \kappa \operatorname{Id}$ with $\kappa \gg 1$. If $\partial_\xi f(0, w_0) \neq 0$ then we may choose $\operatorname{Re} w_2(0)$ so that

$$(5.8) \quad \partial_x f(0, w_0) + \partial_\xi f(0, w_0) \operatorname{Re} w_2(0) = 0$$

Then we find $x'_0(0) = 0$ and we obtain

$$(5.9) \quad \operatorname{Im} w''_0(0) = c - \partial_\xi f(0, w_0) \operatorname{Im} w_2(0) \partial_\xi f(0, w_0) > c/2 > 0$$

by choosing $\operatorname{Im} w_2(0) = \kappa \operatorname{Id}$ with $0 < \kappa \ll 1$. Thus in both cases we find that $\partial_t f(t, w(t)) = \operatorname{Im} w'_0(t) < 0$ at $t = 0$.

We find that $t \mapsto f(t, w(t))$ changes sign from $+$ to $-$ of order k as t increases at $t = 0$. We may then rewrite the equation as

$$(5.10) \quad \operatorname{Im} w'_0(t) = -t^k c(t)$$

where $c(t) > 0$ in a neighborhood of the origin. Since $\operatorname{Im} w_2(0) > 0$ we find that

$$(5.11) \quad e^{i\lambda\omega(t,x)} \leq e^{-c_0\lambda(t^{k+1}+|x-x_0|^2)} \quad |x-x_0| \ll 1 \quad |t| \ll 1$$

Thus the errors that are $\mathcal{O}(|x-x_0|^M)$ in the eikonal equation will give terms that are bounded by $C_M \lambda^{-M/2}$.

We shall also consider the case when $t \mapsto f(t, x, \xi)$ changes sign from $+$ to $-$ of infinite order near Γ . If Γ is not a point, then by reducing to a minimal bicharacteristic as in Section 3, we may assume that $f(t, x_0, \xi_0)$ vanishes of infinite order when $t \in I$ and $|I| \neq 0$. We then obtain an approximate solution to the eikonal equation by solving (4.7)–(4.9) with initial data $w = (x, \xi)$ and $w_k(0)$, $k \geq 2$, which gives a change of coordinates $(t, w) \mapsto (t, w(t))$. If in any neighborhood of $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$ there exist points in $f^{-1}(0)$ where $\partial_t f < 0$, then as before we can construct approximate solutions in any neighborhood of Γ satisfying (5.11) with $k = 1$. If $\partial_t f \geq 0$ on $f^{-1}(0)$ in some neighborhood of Γ , then by the invariance of condition (Ψ) there will still exist a change of sign of $t \mapsto f(t, w(t))$ from $+$ to $-$ in any neighborhood of Γ after the change of coordinates, see [11, Lemma 26.4.11]. (Recall that conditions (2.8) and (2.11) hold in some neighborhood of Γ .) Thus if $F'(t) = -\operatorname{Im} w'_0(t) = f(t, w(t))$ then $t \mapsto F(t)$ has a local maximum at some $t = t_0$, and after subtraction the maximum can be assumed to be equal to 0. By choosing suitable initial value (x_0, ξ_0) for (4.7) at $t = t_0$ we obtain that

$$(5.12) \quad e^{i\lambda\omega(t,x)} \leq e^{\lambda(F(t)-c|x-x_0|^2)} \quad |x-x_0| \ll 1$$

where $F'(t) = f(t, w(t))$ so that $\max_I F(t) = 0$ with $F(t) < 0$ when $t \notin I$ near ∂I .

Proposition 5.2. *Assume that $t \mapsto f(t, x_0, \xi_0)$ changes sign from $+$ to $-$ as t increases near I and that $\partial_t^k \partial_x^\alpha \partial_\xi^\beta f(t, x_0, \xi_0) = 0$ for all $t \in I$ when $|I| \neq 0$. Then we may solve (4.1) in a neighborhood Ω of $\Gamma = \{(t, x_0, \xi_0) : t \in I\}$ modulo $\mathcal{O}(|x-x_0(t)|^M)$, $\forall M$, with*

$\omega(t, x)$ given by (4.2) such that the curve $t \mapsto (x_0(t), \xi_0(t))$, $t \in (t_1, t_2)$, is arbitrarily close to Γ , $w_k(t) \in C^\infty$, $\text{Im } w_2(t) \geq c > 0$ when $t \in (t_1, t_2)$, $\min_{(t_1, t_2)} \text{Im } w_0(t) = 0$ and $\text{Im } w_0(t_j) = c > 0$, $j = 1, 2$.

Observe that since $\text{Im } w_0 \geq 0$ we find that $f(t_0, x_0(t_0), \xi_0(t_0)) = -\text{Im } w'_0(t) = 0$ at a minimum $t_0 \in (t_1, t_2)$. As before, the errors that are $\mathcal{O}(|x - x_0|^M)$ in the eikonal equation will give terms that are bounded by $C_M \lambda^{-M/2}$, $\forall M$. Observe that cutting off where $\text{Im } w_0 > 0$ will give errors that are $\mathcal{O}(\lambda^{-M})$, $\forall M$.

6. THE TRANSPORT EQUATIONS

Next, we shall solve the transport equations given by the following terms in (3.15):

$$(6.1) \quad D_t \phi + \partial_\eta^2 B(t, x, y, \partial_{t,x} \omega, 0) D_y^2 \phi / 2 + A(t, x, y, \partial_{t,x} \omega, 0) D_y \phi + i \partial_\xi f(t, x, \partial_x \omega) D_x \phi \\ + \sum_{j \geq 0} \lambda^{-j} R_j(t, x, y, D_{t,x,y}) \phi$$

near $\Gamma = \{(t, x_0, y_0, 0, \xi_0, 0) : t \in I\}$. Here $R_0(t, x, y) = F_0(t, x, y, \partial_{t,x} \omega, 0)$ and when $m > 0$ we have that $R_m(t, x, y, D_{t,x,y})$ are differential operators of order j in t , order k in x and order ℓ in y , where $j + k + \ell \leq m + 2$. Assuming the conclusions in Proposition 5.2 hold, we shall choose suitable initial values of the amplitude ϕ at $t = t_0$, which is chosen so that $\text{Im } w_0(t_0) = 0$. Observe that the second order differential operator given by the first four terms in (6.1) need not be solvable in general. Instead, by Lemma 6.1 we can treat the D_x and D_y terms as perturbations, using condition (3.6) in the infinite vanishing case.

Since the phase function $\omega(t, x)$ is complex valued, we will replace the values of the symbols at $(\tau, \xi) = \partial_{t,x} \omega(t, x)$ by finite Taylor expansions at $(\text{Re } w'_0(t), \xi_0(t))$. By (4.3) and (4.4) this will give expansions in powers of $x - x_0(t)$ and $\text{Im } w'_0(t) = -f(t, x_0(t), \xi_0(t))$. Then, we shall solve the transport equations up to arbitrarily high powers of $x - x_0(t)$ and f . Since the imaginary part of the phase function $\text{Im } \omega \geq 0$ vanishes of second order at $x = x_0(t)$ we will obtain by Lemma 6.1 below that this will give a solution modulo any negative power of λ .

We shall use the amplitude expansion

$$(6.2) \quad \phi(t, x, y) = \sum_{k \geq 0} \varrho^{-k} \phi_k(t, x, y)$$

and solve the transport equation recursively in k . Here ϕ_k depends on ϱ but with uniform bounds in a suitable symbol class, and $\varrho = \lambda^{1/N}$ with N to be determined later. By doing the change of variables $(t, x, y) \mapsto (t - t_0, x - x_0(t), y - y_0)$ we find that D_t changes into $D_t - x'_0(t) D_x$ which does not change the order of R_j as differential operator. Thus we may assume $t_0 = 0$, $x_0(t) \equiv 0$ and $y_0 = 0$.

Next, we apply (6.1) on ϕ given by (6.2). Since $\varrho = \lambda^{1/N}$ we obtain the terms

$$(6.3) \quad D_t \phi + A_0(t, x) D_x \phi + A_1(t, x, y) D_y \phi + A_2(t, x, y) D_y^2 \phi \\ + \sum_{j \geq 0} \varrho^{-jN} R_j(t, x, y, D_{t,x,y}) \phi$$

where $A_0(t, x) = i\partial_\xi f(t, x, \xi_0(t) + \sigma(t, x)) - x'_0(t)$,

$$(6.4) \quad A_1(t, x, y) = A(t, x, y, \partial_t \omega(t, x), \xi_0(t) + \sigma(t, x), 0)$$

and

$$(6.5) \quad A_2(t, x, y) = \partial_\eta^2 B_2(t, x, y, \partial_t \omega(t, x), \xi_0(t) + \sigma(t, x), 0)/2$$

Here $\sigma(t, x)$ is given by (4.4) and $\partial_t \omega(t, x)$ by (4.3), where the expansion will be up to a sufficiently high order in x . Observe that after the change of variables we have $\sigma(t, 0) \equiv 0$. The values of the symbols will as before be defined by finite Taylor expansions in the τ and ξ variables, which gives expansions in powers of x and $f(t, 0, \xi(t))$.

We are going to construct solutions $\phi_k(t, x, y) = \varphi_k(t, x, \varrho y)$ so that $y \mapsto \varphi_k(t, x, y) \in C_0^\infty$ uniformly in ϱ , which gives localization in $|y| \lesssim \varrho^{-1}$. Therefore we shall choose ϱy as new y coordinates, then (6.3) becomes

$$(6.6) \quad D_t \phi + A_0(t, x) D_x \phi + \varrho A_1(t, x, y/\varrho) D_y \phi + \varrho^2 A_2(t, x, y/\varrho) D_y^2 \phi \\ + \sum_{j \geq 0} \varrho^{-jN} R_j(t, y/\varrho, x, D_t, D_x, \varrho D_y) \phi$$

By Proposition 5.2 the phase function $e^{i\lambda w(t, x)}$ gives the cut-off in x , and we shall expand the symbols in powers of x . Now the Taylor expansion of $x \mapsto \varrho^2 A_2(t, x, y/\varrho)$ will give terms that are $\mathcal{O}(\varrho^2 x)$. Therefore we take $\varrho^2 x$ as new x coordinates, which gives

$$(6.7) \quad D_t \phi + \varrho^2 A_0(t, x/\varrho^2) D_x \phi + \varrho A_1(t, x/\varrho^2, y/\varrho) D_y \phi + \varrho^2 A_2(t, x/\varrho^2, y/\varrho) D_y^2 \phi \\ + \sum_{j \geq 0} \varrho^{-jN} R_j(t, y/\varrho, x/\varrho^2, D_t, \varrho^2 D_x, \varrho D_y) \phi$$

Now the phase function $e^{i\lambda w(t, x)} = \mathcal{O}(e^{-c\varrho^{N-4}|x|^2})$ in the new coordinates. So if we take $N > 4$ it suffices to solve the transport equation up to a sufficiently high order of x , then we may cut off where $|x| \lesssim 1$, which corresponds to $|x| \lesssim \varrho^{-2}$ in the original coordinates. Thus we expand in x :

$$(6.8) \quad \phi_k(t, x, y) = \sum_{k, \alpha} \phi_{k, \alpha}(t, y) x^\alpha \quad \phi_{k, \alpha}(t, y) \in C_0^\infty$$

$$A_0(t, x/\varrho^2) D_x = \sum_{\alpha, j} A_{0, \alpha, j}(t) \varrho^{-2|\alpha|} x^\alpha D_{x_j},$$

$$A_j(t, x/\varrho^2, y/\varrho) = \sum_{\alpha} A_{j, \alpha}(t, y/\varrho) \varrho^{-2|\alpha|} x^\alpha \quad j > 0$$

and

$$(6.9) \quad R_k(t, x/\varrho^2, y/\varrho, \varrho D_y, \varrho^2 D_x) = \sum_{\alpha, \ell, \nu, \mu} R_{k, \alpha, \ell, \nu, \mu}(t, y/\varrho) \varrho^{-2|\alpha|+2|\nu|+|\mu|} x^\alpha D_t^\ell D_x^\nu D_y^\mu$$

Here $\ell + |\nu| + |\mu| \leq k + 2$ so we have at most the factor $\varrho^{2|\nu|+|\mu|} \leq \varrho^{2k+4}$ in (6.9). When $k = 0$ we have $\ell + |\nu| + |\mu| = 0$ and $R_0(t, x/\varrho^2, y/\varrho) = \sum_{\alpha} R_{0, \alpha}(t, y/\varrho) \varrho^{-2|\alpha|} x^\alpha$. Observe that the coefficients in the expansions are given by expansions in powers of $f(t, 0, \xi(t))$. After cut-off in x we find in the original coordinates that $\phi_k(t, x, y) = \varphi_k(t, \varrho^2 x, \varrho y)$ where φ_k for any t is uniformly bounded in C_0^∞ .

We shall first apply (6.7) on ϕ_0 and expand in x . Then we find that the terms that are independent of x are

$$(6.10) \quad D_t \phi_{0,0} - i\varrho^2 \sum_j A_{0,0,j}(t) \phi_{0,e_j} + \varrho A_{1,0}(t, y/\varrho) D_y \phi_{0,0} \\ + \varrho^2 A_{2,0}(t, y/\varrho) D_y^2 \phi_{0,0} + R_{0,0}(t, y/\varrho) \phi_{0,0}$$

We shall need the following result, which gives estimates on f and A_j on the interval of integration. It will be proved in the next section. In the following, we shall denote $f(t) = f(t, 0, \xi_0(t))$ and $F(t) = \int_0^t f(s) ds$. Observe that $f(0) = 0$ since $\text{Im } w'_0(0) = 0$.

Lemma 6.1. *Assume that the conclusions in Proposition 5.2 hold and that (3.6) holds if $t \mapsto f(t)$ vanishes of infinite order at 0. Then there exists ε and $C \geq 1$ with the property that if $N \geq C$, $\varrho = \lambda^{1/N} \geq C$ and*

$$(6.11) \quad |f(t)| + \left| \int_0^t |A_0(s, 0)| + |A_1(s, 0, y/\varrho)| + \|A_2(s, 0, y/\varrho)\| ds \right| \geq C/\varrho^3$$

holds for some $|y| \leq \varrho/C$, then $\lambda F(s) \leq -\lambda^\varepsilon/C$ for some s in the interval connecting 0 and t .

Observe that if Lemma 6.1 holds for some ε and C , then it trivially holds for smaller ε and larger C . We shall assume that $\varepsilon < 1$ and that both N and λ are large enough so that the conclusion in Lemma 6.1 holds. Since (6.11) does not hold when $t = 0$, we can choose the maximal interval I containing 0 such that (6.11) does not hold in I , thus

$$(6.12) \quad |f(t)| + \left| \int_0^t |A_0(s, 0)| + |A_1(s, 0, y/\varrho)| + \|A_2(s, 0, y/\varrho)\| ds \right| < C/\varrho^3 \quad t \in I$$

when $|y| \leq \varrho/C$. By definition we obtain that (6.11) holds for some $|y| \leq \varrho/C$ when $t \in \partial I$ so Lemma 6.1 gives that $\lambda F \lesssim -\lambda^\varepsilon$ at ∂I_0 for some open interval $I_0 \subseteq I$ that contains 0. This means that $e^{i\lambda\omega(t,0)} = e^{\lambda F(t)} \leq C_N \lambda^{-N}$ for any N at ∂I_0 when $\lambda \gg 1$. Since $F' = f$ is uniformly bounded and the left hand side of (6.12) is Lipschitz continuous, we may cut off near I_0 with $\chi(t) \in S(1, \lambda^{6/N} dt^2) \subset S(1, \lambda^{2-2\varepsilon} dt^2)$ for $N \gg 1$ so that $\chi(0) \neq 0$, $\lambda F(t) \lesssim -\lambda^\varepsilon$ in $\text{supp } \chi'$ and (6.12) holds with some C when $t \in \text{supp } \chi$ and $|y| \leq \varrho/C$. Then as before the cut-off errors can be absorbed by the exponential and the expansion in powers of $f(t, 0, \xi(0)) = f(t)$ is justified. In fact, $f(t) = \mathcal{O}(\varrho^{-3})$ in $\text{supp } \chi$, which gives errors of any negative power of $\varrho = \lambda^{1/N}$. The bound on the integral in (6.12) means that we can ignore the A_j terms in (6.10) in $\text{supp } \chi$ modulo lower order terms in ϱ . In the following we shall change the notation and let $I = \text{supp } \chi$. We need to measure the error terms in the following way.

Definition 6.2. For $a(t) \in L^\infty(\mathbf{R})$ and $\kappa > 0$ we say that $a(t) \in I(\kappa)$ if the integral $\int_0^t a(s) ds = \mathcal{O}(\kappa)$ for all $t \in I$.

For example, $f(t) \in I(\varrho^{-3})$ and since the integral in (6.12) is $\mathcal{O}(\varrho^{-3})$ in I the integrand is in $I(\varrho^{-3})$. Then according to (6.12) it suffices to solve

$$(6.13) \quad D_t \phi_{0,0} = -R_{0,0} \phi_{0,0} \quad t \in I$$

to obtain that the terms in (6.10) are in $I(\varrho^{-1})$, here $R_{0,0}(t, y/\varrho) \in C^\infty$ uniformly since $\varrho \geq 1$. Now we can solve (6.13) with $\phi_{0,0}(0, y) = \phi(y) \in C_0^\infty$ uniformly with support where $|y| \ll 1$ such that $\phi(0) = 1$. In fact, the solution is $\phi_{0,0}(t, y) = E(t, y)\phi(y)$, where

$$E(t, y) = \exp \left(-i \int_0^t R_{0,0}(s, y/\varrho) ds \right) \quad t \in I$$

is uniformly bounded in C^∞ . Thus $\phi_{0,0}(t, y) \in C^\infty$ uniformly and by choosing $\phi(y)$ with sufficiently small support we obtain for any $t \in I$ that $\phi_{0,0}(t, \cdot)$ has support in a sufficiently small compact set in which (6.12) holds.

The coefficients of the terms in (6.7) which are homogeneous in x of degree $\alpha \neq 0$ in x are

$$(6.14) \quad D_t \phi_{0,\alpha} + R_{0,0}(t, y/\varrho) \phi_{0,\alpha} - i \sum_{\substack{|\beta|=1 \\ j}} A_{0,\beta,j}(t) (\alpha_j + 1 - \beta_j) \phi_{0,\alpha+e_j-\beta} \\ + \sum_{|\beta|=1} A_{2,\beta}(t, y/\varrho) D_y^2 \phi_{0,\alpha-\beta}$$

modulo $I(\varrho^{-1})$. Letting $\Phi_{k,j} = \{\phi_{k,\alpha}\}_{|\alpha|=j}$ and $\Phi_k = \{\Phi_{k,j}\}_j$ for $k, j \geq 0$, we find that (6.14) vanishes if Φ_0 satisfies the system

$$(6.15) \quad D_t \Phi_{0,k} = S_{0,0}^k \Phi_{0,k} + S_{0,1}^k \Phi_{0,k-1}$$

where $S_{0,0}^k(t)$ is a uniformly bounded matrix depending on t , and $S_{0,1}^k(t, y/\varrho, D_y)$ is a system of uniformly bounded differential operators of order 2 in y when $|y| \lesssim \varrho$. Let $E_{0,k}(t)$ be the fundamental solution to $D_t E_{0,k} = S_{0,0}^k E_{0,k}$ so that $E_{0,k}(0) = \text{Id}$, then letting $\Phi_{0,k}(t, y) = E_{0,k}(t) \Psi_{0,k}(t, y)$ the system (6.15) reduces to

$$D_t \Psi_{0,k}(t, y) = E_{0,k}^{-1} S_{0,1}^k E_{0,k} \Psi_{0,k-1}(t, y)$$

This is a recursion equation which we can solve uniformly in I with $\Psi_{0,k}(t, y)$ having initial values $\Psi_{0,k}(0, y) \equiv 0$ for $0 < k \leq M$. Observe that since the initial data $\Phi_{0,k}(0, y)$ has compact support, we find that $\Phi_{0,k}(t, y) \in C^\infty$ uniformly. For any t we find that $\Phi_{0,k}(t, y)$ has support in a sufficiently small compact set so that (6.12) holds for any $t \in I$.

We shall now apply (6.7) to ϕ given by the full expansion (6.8). We find that the coefficients of the terms in (6.7) which are homogeneous of degree $\alpha \neq 0$ in x are equal to

$$(6.16) \quad \varrho^{-1} (D_t \phi_{1,\alpha} + R_{0,0}(t, y/\varrho) \phi_{1,\alpha} - i \sum_{\substack{|\beta|=1 \\ j}} A_{0,\beta,j}(t) (\alpha_j + 1 - \beta_j) \phi_{1,\alpha+e_j-\beta} \\ + \sum_{|\beta|=1} A_{2,\beta}(t, y/\varrho) D_y^2 \phi_{1,\alpha-\beta} + \sum_{|\beta|=1} A_{1,\beta}(t, y/\varrho) D_y^2 \phi_{0,\alpha-\beta} \\ - i \varrho^3 \sum_j A_{0,0,j}(t) (\alpha_j + 1) \phi_{0,\alpha+e_j} + \varrho^3 A_{2,0}(t, y/\varrho) D_y^2 \phi_{0,\alpha})$$

modulo $I(\varrho^{-2})$. We find that (6.16) vanishes if Φ_1 satisfies the system

$$(6.17) \quad D_t \Phi_{1,k} = S_{1,0}^k \Phi_{1,k} + S_{1,1}^k \Phi_{1,k-1} + \mathbb{A}_1^0 \Phi_0$$

where $S_{1,0}^k(t)$ is a uniformly bounded matrix depending on t , $S_{1,1}^k(t, y/\varrho, D_y)$ is a system of uniformly bounded differential operators of order 2 when $|y| \lesssim \varrho$ and \mathbb{A}_1^0 is a differential operator in y of order 2 with coefficients in $I(1)$ because of (6.12). By letting $\Phi_{1,k} = E_{1,k}\Psi_{1,k}$ with the fundamental solution $E_{1,k}$ to $D_t E_{1,k} = S_{1,0}^k E_{1,k}$, $E_{1,k}(0) = \text{Id}$, this reduces to the equation

$$D_t \Psi_{1,k} = E_{1,k}^{-1} S_{1,1}^k E_{1,k-1} \Psi_{1,k-1} + E_{1,k}^{-1} \mathbb{A}_1^0 \Phi_0$$

Thus we can solve (6.17) in I recursively with uniformly bounded $\Phi_{1,k}$ having initial values $\Phi_{1,k}(0, y) \equiv 0$, $k \geq 0$. But observe that Φ_1 is not in C^∞ uniformly, instead we have $D_t^j \Phi_1 = \mathcal{O}(\varrho^3)$ if $j \geq 1$, since $|\partial_t^j \mathbb{A}_1^0| \leq C_j \varrho^3$, $\forall j$ by (6.16). For that reason, we shall define $S_\varrho^3 \subset C^\infty$ by

$$(6.18) \quad |\partial_t^j \partial_y^\alpha \phi(t, y)| \leq C_{j,\alpha} \varrho^{3j} \quad \forall j, \alpha$$

when $\phi \in S_\varrho^3$. Observe that $\phi \in S_\varrho^3$ if and only if $\phi(t, y) = \chi(\varrho^3 t, y)$ where $\chi \in C^\infty$ uniformly, and that the operator $\varrho^{-3} D_t$ maps $S_\varrho^3 \mapsto S_\varrho^3$. Note that the expansion of the symbols also contains terms with factors $\varrho^3 f^k$, $k \geq 1$, which are uniformly bounded in S_ϱ^3 for $t \in I$ by (6.12). Since $\int_0^t \mathbb{A}_1^0 dt \in S_\varrho^3$ in I we find that $\Phi_1 \in S_\varrho^3$ in I .

Recursively, the coefficients of the terms in (6.7) that are homogeneous in x of degree α are

$$(6.19) \quad \begin{aligned} & \varrho^{-k} (D_t \phi_{k,\alpha} - i \sum_{\beta \neq 0} A_{0,\beta,j}(t) (\alpha_j + 1 - \beta_j) \phi_{k+2-2|\beta|,\alpha+e_j-\beta} \\ & + \sum_{\beta \neq 0} A_{1,\beta}(t, y/\varrho) D_y \phi_{k+1-2|\beta|,\alpha-\beta} + \sum_{\beta \neq 0} A_{2,\beta}(t, y/\varrho) D_y^2 \phi_{k+2-2|\beta|,\alpha-\beta} \\ & - i \varrho^3 \sum_j A_{0,0,j}(t) (\alpha_j + 1) \phi_{k-1,\alpha+e_j} + \varrho^3 A_{1,0}(t, y/\varrho) D_y \phi_{k-2,\alpha} \\ & + \varrho^3 A_{2,0}(t, y/\varrho) D_y^2 \phi_{k-1,\alpha} \\ & + \sum_{\ell+|\nu|+|\mu| \leq j+2} \varrho^{-jN} R_{j,\beta,\ell,\nu,\mu}(t, y/\varrho) c_{\alpha,\beta,\nu} \varrho^{-2|\beta|+2|\nu|+|\mu|+i+3\ell} (\varrho^{-3} D_t)^\ell D_y^\mu \phi_{k-i,\alpha+\nu-\beta} \end{aligned}$$

modulo $I(\varrho^{-k-1})$. Here the last sum has $\ell + |\nu| + |\mu| = 0$ when $j = 0$, $(\varrho^{-3} D_t)^\ell D_y^\mu$ maps $S_\varrho^3 \mapsto S_\varrho^3$ and the values of the symbols are given by a finite expansion in powers of $f(t)$.

Since $\phi_j \in S_\varrho^3$ we obtain that the terms in (6.19) are in $I(\varrho^{-k-1})$ if

$$(6.20) \quad \begin{aligned} & D_t \phi_{k,\alpha} - i \sum_{\beta \neq 0} A_{0,\beta,j}(t) (\alpha_j + 1 - \beta_j) \phi_{k+2-2|\beta|,\alpha+e_j-\beta} \\ & + \sum_{\beta \neq 0} A_{1,\beta}(t, y/\varrho) D_y \phi_{k+1-2|\beta|,\alpha-\beta} + \sum_{\beta \neq 0} A_{2,\beta}(t, y/\varrho) D_y^2 \phi_{k+2-2|\beta|,\alpha-\beta} \\ & - i \varrho^3 \sum_j A_{0,0,j}(t) (\alpha_j + 1) \phi_{k-1,\alpha+e_j} + \varrho^3 A_{1,0}(t, y/\varrho) D_y \phi_{k-2,\alpha} \\ & + \varrho^3 A_{2,0}(t, y/\varrho) D_y^2 \phi_{k-1,\alpha} \\ & = - \sum_{\substack{i+3\ell+2|\nu|+|\mu|=jN+2|\beta| \\ \ell+|\nu|+|\mu| \leq j+2}} R_{j,\beta,\ell,\nu,\mu}(t, y/\varrho) c_{\alpha,\beta,\nu} (\varrho^{-3} D_t)^\ell D_y^\mu \phi_{k-i,\alpha+\nu-\beta} \end{aligned}$$

When $j = 0$ we find that $\ell + |\nu| + |\mu| = 0$, $i = 2|\beta|$ and we only have an expansion in β in the last sum. Now if $j > 0$, $\ell + |\nu| + |\mu| \leq j + 2$ and $i + 3\ell + 2|\nu| + |\mu| = jN + 2|\beta|$ then we find that

$$jN \leq i + 3\ell + 2|\nu| + |\mu| < i + 3(j + 2)$$

which gives $i \geq j(N - 3) - 6 \geq N - 9 \geq 1$ if $N \geq 10$. Thus we find that (6.20) can be written as

$$(6.21) \quad D_t \Phi_k = \mathbb{A}_0^k \Phi_k + \mathbb{A}_1^k \Phi_{k-1} + \mathbb{A}_2^k \Phi_{k-2} + \dots$$

where $\int_0^t \mathbb{A}_j^k dt$ is a uniformly bounded differential operator on S_ϱ^3 for $t \in I$ and $j > 0$. We have that

$$\{ \mathbb{A}_0^k \Phi_k \}_j = S_{k,0}^j \Phi_{k,j} + S_{k,1}^j \Phi_{k,j-1}$$

where $S_{k,0}^j(t)$ is a uniformly bounded matrix depending on t , and $S_{k,1}^j(t, y/\varrho, D_y)$ is a system of uniformly bounded differential operators of order 2 when $|y| \lesssim \varrho$. By letting $\Phi_{k,j} = E_{k,j} \Psi_{k,j}$ with the fundamental solution $E_{k,j}$ to $D_t E_{k,j} = S_{k,0}^j E_{k,j}$, $E_{k,j}(0) = \text{Id}$, (6.21) becomes a system of recursion equations in j and k . Thus (6.21) can be solved in I with $\Phi_k \in S_\varrho^3$ having initial values $\Phi_k(0) \equiv 0$, $k > 0$. We find from (6.8) and the definition of S_ϱ^3 that $\phi_k(t, x, y) = \varphi_k(\varrho^3 t, \varrho^2 x, \varrho y)$ where $\varphi_k \in C^\infty$ uniformly when $t \in I$. Thus we can solve the transport equation (6.1) up to any negative power of λ . Observe that by cutting off in t and x we may assume that $\varphi_k \in C_0^\infty$ has fixed compact support in (x, y) and support where $|t| \lesssim \varrho^3$. It follows that the support of ϕ_k can be chosen in an arbitrarily small neighborhood of Γ for large enough λ . Changing to the original coordinates, we obtain the following result.

Proposition 6.3. *Assume that the conclusions in Proposition 5.2 hold, and that (3.6) is satisfied near Γ when the sign change of $t \mapsto f(t, x_0, \xi_0)$ is of infinite order. If $\varrho = \lambda^{1/N}$ for sufficiently large N , then for any K and M we can solve the transport equations (6.20) for $k \leq K$ and $|\alpha| \leq M$ near $\{ (t, x_0(t), y_0) : t \in [t_1, t_2] \}$. By (6.8) this gives*

$$\phi_k(t, x, y) = \varphi_k(\varrho^3(t - t_0), \varrho^2(x - x_0(t)), \varrho(y - y_0)) \quad k \leq K$$

where $\varphi_k(t, x, y) \in C^\infty$ uniformly, has support where $|x| + |y| \lesssim 1$ and $|t| \lesssim \varrho^3$, and $\varphi_0(0, 0, 0) = 1$ for some $t_0 \in (t_1, t_2)$ such that $\text{Im } w_0(t_0) = 0$.

7. THE RATE OF CHANGE OF SIGN

We have showed that $t \mapsto f(t, x, \xi)$ changes sign from $+$ to $-$ on an interval I . Then

$$(7.1) \quad F(t) = \int_0^t f(s, x_0(s), \xi_0(s)) ds = \int_0^t f(s) ds$$

has a local maximum in the interval. By choosing that maximum as the starting point we may assume it is equal to 0 so that $F(t) \leq 0$. By changing t coordinate, we may assume $F(0) = 0$. We shall study how the size of the derivative f affects the size of the function F .

Lemma 7.1. *Assume that $0 \geq F(t) \in C^\infty$ has local maximum at $t = 0$, and let I_{t_0} be the closed interval joining 0 and $t_0 \in \mathbf{R}$. If*

$$\max_{I_{t_0}} |F'(t)| = |F'(t_0)| = \kappa \leq 1$$

with $|t_0| \geq \kappa^\varrho$ for some $\varrho > 0$, then we have $\min_{I_{t_0}} F(t) \leq -C_\varrho \kappa^{1+\varrho}$. The constant $C_\varrho > 0$ only depends on ϱ and the bounds on F in C^∞ .

Proof. Let $f = F'$ then since $F(t) = F(0) + \int_0^t f(s) ds \leq \int_0^t f(s) ds$ it is no restriction to assume the maximum $F(0) = 0$. By switching t to $-t$ we may assume $t_0 \leq -\kappa^\varrho < 0$. Let

$$(7.2) \quad g(t) = \kappa^{-1} f(t_0 + t\kappa^\varrho)$$

then $|g(0)| = 1$, $|g(t)| \leq 1$ for $0 \leq t \leq 1$ and

$$|g^{(N)}(t)| = \kappa^{\varrho N-1} |f^{(N)}(t_0 + t\kappa^\varrho)| \leq C_N$$

when $N \geq 1/\varrho$ for $0 \leq t \leq 1$. By using the Taylor expansion at $t = 0$ for $N \geq 1/\varrho$ we find

$$(7.3) \quad g(t) = p(t) + r(t)$$

where p is the Taylor polynomial of order $N - 1$ of g at 0, and

$$(7.4) \quad r(t) = t^N \int_0^1 g^{(N)}(ts) (1-s)^{N-1} ds / (N-1)!$$

is uniformly bounded in C^∞ for $0 \leq t \leq 1$ and $r(0) = 0$. Since g also is bounded on the interval, we find that $p(t)$ is uniformly bounded in $0 \leq t \leq 1$. Since all norms on the finite dimensional space of polynomials of fixed degree are equivalent, we find that $p^{(k)}(0) = g^{(k)}(0)$ are uniformly bounded for $0 \leq k < N$ which implies that $g(t)$ is uniformly bounded in C^∞ for $0 \leq t \leq 1$. Since $|g(0)| = 1$ it exists a uniformly bounded $\delta^{-1} \geq 1$ such that $|g(t)| \geq 1/2$ when $0 \leq t \leq \delta$, thus g has the same sign in that interval. Since $g(s) = \kappa^{-1} f(t_0 + s\kappa^\varrho)$ we find

$$(7.5) \quad \delta/2 \leq \left| \int_0^\delta g(s) ds \right| = \left| \kappa^{-\varrho} \int_{t_0}^{t_0+\delta\kappa^\varrho} \kappa^{-1} f(t) dt \right|$$

Since $t_0 + \delta\kappa^\varrho \leq 0$ we find that the variation of $F(t)$ on $[t_0, 0]$ is greater than $\delta\kappa^{1+\varrho}/2$ and since $F \leq 0$ we find that the minimum of F on I_{t_0} is smaller than $-\delta\kappa^{1+\varrho}/2$. \square

Proof of Lemma 6.1. As before we let $F(t)$ satisfy $F(0) = 0$ and $F'(t) = f(t)$ where $f(t) = f(t, 0, \xi_0(t))$ satisfies $f(0) = 0$. We have assumed that the estimate (3.6) holds near Γ if $f(t)$ vanishes of infinite order at $t = 0$. Observe that the term $x'_0(t)$ in A_0 can be estimated by $|\partial_w f(t, 0, \xi_0(t))|$ by (4.7), which gives that $|A_0(t, 0)| \lesssim |\partial_w f(t, 0, \xi_0(t))|$. We find from (4.3), (4.6) and (4.7) that $|\partial_t \omega(t, 0)| \lesssim |f(t)| + |\partial_w f(t, 0, \xi_0(t))|$ thus (6.11) follows if

$$(7.6) \quad |f(t)| + \left| \int_0^t |f(s)| + A_0(s, 0) + A_1(s, 0, y/\varrho) + A_2(s, 0, y/\varrho) ds \right| \gtrsim \varrho^{-3}$$

where now $A_0(t) = |\partial_w f(t, 0, \xi_0(t))|$,

$$(7.7) \quad A_1(t, y/\varrho) = |A(t, 0, y/\varrho, 0, \xi_0(t), 0)|$$

and

$$(7.8) \quad A_2(t, y/\varrho) = \|\partial_\eta^2 B(t, 0, y/\varrho, 0, \xi_0(t), 0)\|$$

In the following we shall suppress the y variables in (7.6), the results will be uniform when $|y| \leq c\varrho$ for some $c > 0$ since (3.6) holds near Γ . Observe that if $|f(s)|$ and $|A_j(s)|$ are $\ll \varrho^{-3}$ for $0 \leq j \leq 2$ when s is between 0 and t , then (7.6) does not hold.

We shall first consider the case when $|f(t)| \cong |t|^m$ vanishes of finite order at $t = 0$. Then the order must be odd so we find $F(t) = \int_0^t f(s) ds \leq 0$ and $c \leq |F(t)|/t^{2k} \leq C < 0$ for some $k > 0$. Thus we find

$$(7.9) \quad \varrho^{-3} \lesssim \left| \int_0^t |f(s)| + A_0(s) + A_1(s) + A_2(s) ds \right| \lesssim |t| \lesssim |F(t)|^{1/2k}$$

implies that $|F(t)| \gtrsim \varrho^{-6k}$. Since $\lambda = \varrho^N$ we then obtain $\lambda F(t) \lesssim -\varrho^{N-6k} \leq -\varrho = \lambda^{1/N}$ if $N > 6k$. The case when $|t|^{2k-1} \cong |f(t)| \gtrsim \varrho^{-3}$ gives that $|t| \gtrsim \varrho^{-3/2k-1}$ so $\lambda F(t) \lesssim -\varrho^{N-\frac{6k}{2k-1}} \leq -\varrho$ if $N > 6$. Now one of these cases must hold if (6.11) holds, so we get the result in the finite vanishing case.

Next, we consider the infinite vanishing case, then we have assumed that condition (3.6) holds, which means that

$$\sum_{j=0}^2 A_j(t) \lesssim |f(t)|^\varepsilon$$

which implies that $A_j(0) = 0$ for all j . Now we assume that (7.6) holds at t , by switching t and $-t$ we may assume $t > 0$. Then we obtain for some $s \in [0, t]$ that $|f(s)| \geq c\varrho^{-3}$ or $A_j(s) \geq c\varrho^{-3}$ for some $c > 0$ and j . Now we define t_0 as the smallest $t_0 > 0$ such that $|f(t_0)| = c\varrho^{-3}$ or $A_j(t_0) = c\varrho^{-3}$ for some j , then $t_0 \leq t$. Then we obtain from condition (3.6) in the first case that $c\varrho^{-3} = |f(t_0)| \lesssim |f(t_0)|^\varepsilon$ and in the second case that

$$(7.10) \quad c\varrho^{-3} = A_j(t_0) \lesssim |f(t_0)|^\varepsilon$$

Since $\varrho = \lambda^{1/N}$ we find in both cases that

$$(7.11) \quad \lambda^{-3/\varepsilon N} = \kappa \leq c|f(t_0)| \quad c > 0$$

where $\lambda \gg 1$ if and only if $\kappa \ll 1$. By taking the smallest t_0 such that (7.11) is satisfied, we find that $|f(t)| \leq |f(t_0)|$ for $0 \leq t \leq t_0$. Since $f(t)$ vanishes of infinite order at $t = 0$, we find by using Taylor's formula that $|f(t)| \leq C_M |t|^M$ for any positive integer M . (Actually, it suffices to take $M = 1$.) Condition (7.11) then gives

$$(7.12) \quad \kappa^{1/M} \lesssim |f(t_0)|^{1/M} \lesssim |t_0|$$

so by using Lemma 7.1 with $\varrho = 1/M$ we find that

$$(7.13) \quad \min_{0 \leq s \leq t_0} F(s) \lesssim -\kappa^{1+1/M} = -\lambda^{-3(1+1/M)/\varepsilon N} \quad \lambda \gg 1$$

Thus we find that $\min_{0 \leq s \leq t_0} F(s) \lesssim -\lambda^{c-1}$ for some $c > 0$ if $3(1 + 1/M)/\varepsilon N < 1$, i.e., $N > 3(1 + 1/M)/\varepsilon$, which gives Lemma 6.1. \square

8. THE PROOF OF THEOREM 2.7

We shall use the following modification of Lemma 26.4.15 in [11]. Recall that $\|u\|_{(k)}$ is the L^2 Sobolev norm of order k of $u \in C_0^\infty$ and let $\mathcal{D}'_\Gamma = \{u \in \mathcal{D}' : \text{WF}(u) \subset \Gamma\}$ for $\Gamma \subseteq T^*\mathbf{R}^n$.

Lemma 8.1. *Let*

$$(8.1) \quad u_\lambda(x) = \lambda^{(n-1)\delta/2} \exp(i\lambda\omega(x)) \sum_{j=0}^M \varphi_j(\lambda^\delta x) \lambda^{-j\varrho} \quad \lambda \geq 1$$

with $\varrho > 0$, $0 < \delta < 1$, $\omega \in C^\infty(\mathbf{R}^n)$ satisfying $\text{Im } \omega \geq 0$, $|d\text{Re } \omega| \geq c > 0$, and $\varphi_j \in C_0^\infty(\mathbf{R}^n)$. Here ω and φ_j may depend on λ but uniformly, and φ_j has fixed compact support in all but one of the variables, for which the support is bounded by $C\lambda^\delta$. Then for any integer N we have

$$(8.2) \quad \|u_\lambda\|_{(-N)} \leq C\lambda^{-N} \quad \lambda \geq 1$$

If $\varphi_0(x_0) \neq 0$ and $\text{Im } \omega(x_0) = 0$ for some x_0 then there exists $c > 0$ so that

$$(8.3) \quad \|u_\lambda\|_{(-N)} \geq c\lambda^{-N-\frac{n}{2}+\frac{(n-1)\delta}{2}} \quad \lambda \geq 1 \quad \forall N$$

Let $\Sigma = \bigcap_{\lambda \geq 1} \bigcup_j \text{supp } \varphi_j(\lambda^\delta \cdot)$ and let Γ be the cone generated by

$$(8.4) \quad \{ (x, \partial\omega(x)), x \in \Sigma, \text{Im } \omega(x) = 0 \}$$

Then for any k we find $\lambda^k u_\lambda \rightarrow 0$ in \mathcal{D}'_Γ so $\lambda^k A u_\lambda \rightarrow 0$ in C^∞ if A is a pseudodifferential operator such that $\text{WF}(A) \cap \Gamma = \emptyset$. The estimates are uniform if $\omega \in C^\infty$ uniformly with fixed lower bound on $|d\text{Re } \omega|$, and $\varphi_j \in C_0^\infty$ uniformly with the support condition.

In the expansion (8.1) we shall take $\varrho = 1/N$ and $\delta = 3/N$ with $N > 3$, and the cone Γ will be generated by

$$(8.5) \quad \{ (t, x_0(t), y_0, 0, \xi_0(t), 0) : t \in I \}$$

where $I = \{ t : \text{Im } w_0(t) = 0 \}$. Observe that the phase function in (4.2) will satisfy the conditions in Lemma 8.1 near $\{ (t, x_0(t), y_0) : t \in I \}$ since $\xi_0(t) \neq 0$ and $\text{Im } \omega(t, x) \geq 0$ by Proposition 5.2. Also, we find from Proposition 6.3 that the functions φ_k will satisfy the conditions in Lemma 8.1 with $\delta = 3/N$ after making the change of variables $(t, x, y) \mapsto (t - t_0, x - x_0(t), y - y_0)$ since $\varphi_0(t_0, x_0(t_0), y_0) = 1$. Observe that the conclusions of Lemma 8.1 are invariant under uniform changes of coordinates.

Proof of Lemma 8.1. We shall modify the proof of [11, Lemma 26.4.15] to this case. We have that

$$(8.6) \quad \hat{u}_\lambda(\xi) = \lambda^{(n-1)\delta/2} \sum_{j=0}^M \lambda^{-j\delta} \int e^{i\lambda\omega(x) - i\langle x, \xi \rangle} \varphi_j(\lambda^\delta x) dx$$

Let U be a neighborhood of the projection on the second component of the set in (8.4). When $\xi/\lambda \notin U$ then for $\lambda \gg 1$ we find that

$$\bigcup_j \text{supp } \varphi_j(\lambda^\delta \cdot) \ni x \mapsto (\lambda\omega(x) - \langle x, \xi \rangle)/(\lambda + |\xi|)$$

is in a compact set of functions with non-negative imaginary part with a fixed lower bound on the gradient of the real part. Thus, by integrating by parts we find for any positive integer k that

$$(8.7) \quad |\hat{u}_\lambda(\xi)| \leq C_k \lambda^{((n-1)/2+k)\delta} (\lambda + |\xi|)^{-k} \quad \xi/\lambda \notin U \quad \lambda \gg 1$$

which gives any negative power of λ for k large enough, since $\delta < 1$. If V is bounded and $0 \notin \overline{V}$ then since u_λ is uniformly bounded in L^2 we find

$$(8.8) \quad \int_{\lambda V} |\hat{u}_\lambda(\xi)|^2 (1 + |\xi|^2)^{-N} d\xi \leq C_V \lambda^{-2N}$$

which together with (8.7) gives (8.2). If $\chi \in C_0^\infty$ then we may apply (8.7) to χu_λ , thus we find for any positive integer k that

$$(8.9) \quad |\widehat{\chi u_\lambda}(\xi)| \leq C \lambda^{((n-1)/2+k)\delta} (\lambda + |\xi|)^{-k} \quad \xi \in W \quad \lambda \gg 1$$

if W is any closed cone with $(\text{supp } \chi \times W) \cap \Gamma = \emptyset$. Thus we find that $\lambda^k u_\lambda \rightarrow 0$ in \mathcal{D}'_Γ for every k . To prove (8.3) we may assume that $x_0 = 0$ and take $\psi \in C_0^\infty$. If $\text{Im } \omega(0) = 0$ and $\varphi_0(0) \neq 0$ then since $\delta < 1$ we obtain that

$$(8.10) \quad \lambda^{n-(n-1)\delta/2} e^{-i\lambda \text{Re } w(0)} \langle u_\lambda, \psi(\lambda \cdot) \rangle = \int e^{i\lambda(w(x/\lambda) - \text{Re } w(0))} \psi(x) \sum_j \varphi_j(\lambda^{\delta-1} x) \lambda^{-j\delta} dx \\ \rightarrow \int e^{i\langle \text{Re } \partial_x \omega(0), x \rangle} \psi(x) \varphi_0(0) dx \quad \lambda \rightarrow \infty$$

which is not equal to zero for some suitable $\psi \in C_0^\infty$. Since

$$(8.11) \quad \|\psi(\lambda \cdot)\|_{(N)} \leq C_N \lambda^{N-n/2}$$

we obtain from (8.10) that $0 < c \leq \lambda^{N+n/2-(n-1)\delta/2} \|u_\lambda\|_{(-N)}$ which gives (8.3) and the lemma. \square

Proof of Theorem 2.7. By conjugating with elliptic Fourier integral operators and multiplying with pseudodifferential operators, we may obtain that $P^* \in \Psi_{cl}^2$ is on the form given by Proposition 3.2 microlocally near $\Gamma = \{(t, x_0, y_0, 0, \xi_0, 0) : t \in I\}$. Thus we may assume

$$(8.12) \quad P^* = D_t + F(t, x, y, D_t, D_x, D_y) + R$$

where $R \in \Psi_{cl}^2$ satisfies $\text{WF}(R) \cap \Gamma = \emptyset$.

Now we can construct approximate solutions u_λ on the form (3.14) by using the expansion (3.15). By reducing to minimal bicharacteristics, we may solve first the eikonal equation by using Proposition 5.2 and then the transport equations (6.20) by using Proposition 6.3 with $\varrho = \lambda^{1/N}$ for $N > 3$. Thus after making the change of coordinates $(t, x, y) \mapsto (t - t_0, x - x_0(t), y - y_0)$ we obtain approximate solutions u_λ on the form (8.1) in Lemma 8.1 with $\varrho = 1/N$ and $\delta = 3/N$. For N large enough, we may choose K and M in Proposition 6.3 so that $|(D_t + F)u_\lambda| \lesssim \lambda^{-k}$ for any k . Now differentiation of $(D_t + F)u_\lambda$ can at most give a factor λ since $\delta < 1$ and a loss of a factor $x - x_0(t)$ gives at most a factor $\lambda^{1/2}$. Because of the bounds on the support of u_λ we may obtain that

$$(8.13) \quad \|(D_t + F)u_\lambda\|_{(\nu)} = \mathcal{O}(\lambda^{-N-n})$$

for any chosen ν . Since $\varphi_0(t_0, x_0(t_0), y_0) = 1$ by Proposition 6.3 and $\text{Im } w(t_0, x_0(t_0)) = 0$ by Proposition 5.2, we find by (8.2)–(8.3) that

$$(8.14) \quad \lambda^{-N-\frac{n}{2}} \ll \lambda^{-N-\frac{n}{2}+\frac{(n-1)\delta}{2}} \lesssim \|u\|_{(-N)} \lesssim \lambda^{-N} \quad \forall N \quad \lambda \gg 1$$

Since u_λ has support in a fixed compact set that shrinks towards $\{(t, x_0(t), y_0) : t \in I\}$ as $\lambda \rightarrow \infty$, we find from Lemma 8.1 that $\|Ru\|_{(\nu)}$ and $\|Au\|_{(0)}$ are $\mathcal{O}(\lambda^{-N-n})$ if $\text{WF}(A)$ does not intersect Γ . Thus we find from (8.13) and (8.14) that (2.17) does not hold when $\lambda \rightarrow \infty$, so P is not solvable at Γ by Remark 2.10. \square

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